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Hot electrons under quantization conditions: III. Analytical results and new nonlinear regimes

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Abstract. It is shown that the kinetic behaviour of a one-dimensional electron system is qualitatively different at low and high lattice temperatures. At low lattice temperatures the interaction has a strong inelastic character for the majority of electrons. As a result the electron distribution function is to be found from the integro-differential equation. This equation was solved analytically and we obtain the new distribution functions. We have shown that the current–voltage characteristic obeys a sublinear behaviour for warm and hot electrons. Within a wide range of the external electric field E the distribution function for the hot electrons has a sharp anisotropic shape corresponding to the electron streaming regime. The electric-field-dependences of the hot-electron mobility and the mean energy are $E^{-5/6}$ and $E^{1/2}$, respectively. With increasing E the electron–acoustic-phonon interaction becomes quasi-elastic an the electron distribution function, which is quasi-isotropic, is described by a differential equation of the Fokker–Planck type. No runaway effect arises in strong electric fields, the electron mobility does not depend on E (the 'second ohmic regime') and the mean energy increases as E^4. In the opposite case of high lattice temperatures the electron–acoustic-phonon interaction is always quasi-elastic for a majority of the electrons. The scattering rate decreases when the energy of the electron increases. This results in a runaway effect for hot electrons in a quantum wire and superlinear behaviour of the current–voltage characteristic. To stabilize the one-dimensional electron system it is necessary to take into account the transition of electrons to the continuous energy spectrum for thick quantum wires or interaction with optical phonons for thin quantum wires. We have derived general expressions for the distribution functions under different conditions which are of experimental interest. The theory we have developed can be generalized for a two- or three-dimensional electron gas subjected to an arbitrary quantizing potential, as well as to incorporate other scattering mechanisms.

1. Introduction

In [1] we established the basic kinematics of the electron–acoustic-phonon interaction and in [2] we derived the Boltzmann equation for electrons in terms of symmetrical and antisymmetrical components of the distribution function and obtained general expressions for the cases of quasi-elastic and inelastic scattering. In this paper we obtain analytical solutions of the Boltzmann equation for various circumstances, beginning in section 2 with the case of inelastic scattering. There are significant difficulties in solving what is an integro-differential equation and these are discussed in depth. A transformation to an integral equation can be made under certain circumstances and this allows us to consider some analytically solvable cases, beginning with the study of warm electrons in section 3, for which the energy gained from the electric field is small compared with the thermal value. On extending

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the discussion to hot electrons in section 4 we find that the distribution becomes highly anisotropic, corresponding to electron streaming. Hot electrons in the zero-point lattice are treated in section 5, in which it is shown that the electron distribution is stabilized under these conditions. In section 6 we turn to the case of quasi-elastic scattering at high lattice temperatures and consider solutions of the Boltzmann equation when equipartition holds. In this case we obtain runaway conditions. Finally, in section 7, we study the macroscopic characteristics of a 1D hot electron gas. The mathematical details of the calculations are presented in the corresponding appendices. The previous papers [1, 2] hereafter are referred to as I and II, respectively.

2. The solution of the Boltzmann equation for inelastic electron–acoustic-phonon scattering

The electron–acoustic-phonon interaction has an inelastic character for intra-sub-band scattering within every sub-band if the electron kinetic energy in the sub-band is within the range $\varepsilon_\parallel(\kappa_x) < (8m^*s^2W_0)^{1/2}$ (here and below we use the same notation as in the previous papers I and II). In the case in which several sub-bands are involved in the relaxation process, the electrons participate simultaneously in inelastic intra-sub-band and quasi-elastic inter-sub-band scattering. As a result the peculiarities of the inelasticity of the electron–acoustic-phonon interaction are smoothed out. However, if the electrons occupy only the first sub-band and then inelastic scattering is the sole mechanism of relaxation of the symmetrical and antisymmetrical distribution functions. This is the physical reason why we will deal with the case of occupation of the first sub-band ($\nu = \nu' = 1$). It is obvious that the scattering is inelastic for the majority of electrons even in the thermal equilibrium state if the lattice temperature T_0 (which defines also the average electron energy in equilibrium) is within the range given by $T_0 < (8m^*sW_0)^{1/2}$. Therefore, we will investigate the non-equilibrium kinetics of the electron–phonon system under the conditions defined by

$$\varepsilon_\parallel(\kappa_x) < (8m^*s^2W_0)^{1/2} \tag{2.1}$$

$$T_0 < (8m^*s^2W_0)^{1/2} \tag{2.2}$$

when electrons occupy only the first sub-band. It is convenient to use the kinetic energy $\varepsilon_\parallel$ instead of the total energy ε ($\varepsilon_\parallel = \varepsilon - W_1$) everywhere in the arguments of the corresponding functions and to drop the sub-band index for simplicity of notation. Due to equation (2.1) we can put the form factor equal to unity.

The expression for the antisymmetrical collision operator $\hat{I}F^-(\kappa_x)$ was obtained in paper II (see equations (3.4), (3.10) and (3.11) in II). By making use of these equations in the Boltzmann kinetic equation (2.11) in paper II, we obtain the solution for the antisymmetrical distribution function in the usual form

$$F^-(\kappa_x) = \frac{1}{\hbar} eE_x\tau(\varepsilon_\parallel)\frac{\mathrm{d}F_0(\varepsilon_\parallel)}{\mathrm{d}\kappa_x}. \tag{2.3}$$

Here the momentum relaxation time $\tau(\varepsilon_\parallel)$ is given by

$$\frac{1}{\tau(\xi_\parallel)} = \frac{1}{\tau_0}\frac{1}{\Psi(\xi_\parallel)} \tag{2.4}$$

where

$$\frac{1}{\tau_0} = \frac{\Xi_a^2(m^*)^{1/2}T_0^{5/2}}{2\sqrt{2}\pi\rho\hbar^4 s^4} \tag{2.5}$$

$$\frac{1}{\Psi(\xi_\parallel)} = \int_0^\infty \frac{N_\omega}{(\xi_\parallel+\omega)^{1/2}}\omega^2\,\mathrm{d}\omega + \int_0^{\xi_\parallel} \frac{N_\omega+1}{(\xi_\parallel+\omega)^{1/2}}\omega^2\,\mathrm{d}\omega \tag{2.6}$$

$$N_\omega = (\mathrm{e}^\omega - 1)^{-1} \qquad \xi_\parallel = \frac{\varepsilon_\parallel}{T_0} \qquad \omega = \frac{\hbar s q_\perp}{T_0}. \tag{2.7}$$

The expression for the symmetrical collision operator $\hat{I}F_0(\varepsilon_\parallel)$ was obtained in paper II (equation (5.6) in II). Upon inserting the antisymmetrical distribution function $F^-(\kappa_x)$ from equation (2.3) and $\hat{I}F_0(\varepsilon_\parallel)$ into the Boltzmann equation (2.12) in paper II, we obtain an equation for the symmetrical distribution function $F_0(\xi_\parallel)$:

$$-\varepsilon_E^2\xi_\parallel^{1/2}\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left(\xi_\parallel^{1/2}\Psi(\xi_\parallel)\frac{\mathrm{d}F_0(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right) = \int_0^\infty \frac{\omega^2}{(\xi_\parallel+\omega)^{1/2}}[F_0(\xi_\parallel+\omega)(N_\omega+1) - F_0(\xi_\parallel)N_\omega]\,\mathrm{d}\omega$$
$$+\int_0^{\xi_\parallel}\frac{\omega^2}{(\xi_\parallel-\omega)^{1/2}}[F_0(\xi_\parallel-\omega)N_\omega - F_0(\xi_\parallel)(N_\omega+1)]\,\mathrm{d}\omega \tag{2.8}$$

where we have introduced the dimensionless electric field

$$\varepsilon_E^2 = \frac{E_x^2}{E_c^2} \qquad E_c^2 = \frac{m^*T_0}{2\mathrm{e}^2\tau_0^2}. \tag{2.9}$$

Equation (2.8) contains only one parameter, ε_E^2. This is an integro-differential equation and there are significant difficulties in solving such equations [3].

By making use of equation (2.6) and changing the variable of integration we can transform equation (2.8) into the form

$$\varepsilon_E^2\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left(\Psi_1(\xi_\parallel)\frac{\mathrm{d}F_0(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right) - \frac{F_0(\xi_{\parallel})}{\Psi_1(\xi_\parallel)} = -\int_0^\infty K_1(\omega,\xi_\parallel)F_0(\omega)\,\mathrm{d}\omega \tag{2.10}$$

where we have introduced the new notations

$$\Psi_1(\xi_\parallel) = \xi_\parallel^{1/2}\Psi(\xi_\parallel) \tag{2.11}$$

$$K_1(\omega,\xi_\parallel) = \xi_\parallel^{1/2}\,\mathrm{e}^\omega K(\omega,\xi_\parallel) \tag{2.12}$$

$$K(\omega,\xi_\parallel) = \frac{(\xi_\parallel-\omega)^2}{\omega^{1/2}}\left|\frac{1}{\mathrm{e}^{\xi_\parallel}-\mathrm{e}^\omega}\right|. \tag{2.13}$$

Note that the expressions in equation (2.6) can be presented in the form

$$\frac{1}{\Psi(\xi_\parallel)} = \mathrm{e}^{\xi_\parallel}\int_0^\infty K(\omega,\xi_\parallel)\,\mathrm{d}\omega. \tag{2.14}$$

The function $K_1(\omega,\xi_\parallel)$ in the integrand of the right-hand side of equation (2.10) is the kernel of the integro-differential equation.

To find the solution of equation (2.10) we will consider it formally as an inhomogeneous differential equation [4]

$$\hat{D}F_0(\xi_\parallel) = \Phi(\xi_\parallel) \tag{2.15}$$

where $\hat{D}$ is the differential operator,

$$\hat{D} = \varepsilon_E^2\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel} - \frac{1}{\Psi_1(\xi_\parallel)}. \tag{2.16}$$

The inhomogeneous part $\Phi(\xi_\parallel)$ in equation (2.15) is a functional of $F_0(\xi_\parallel)$:

$$\Phi(\xi_\parallel) = -\int_0^\infty K_1(\omega,\xi_\parallel)F_0(\xi_\parallel)\,\mathrm{d}\omega. \tag{2.17}$$

Let us first find the eigenfunctions $\tilde{F}_0(\xi_\parallel)$ of the differential operator $\hat{D}$ with zero eigenvalues:

$$\hat{D}\tilde{F}_0(\xi_\parallel) = 0. \tag{2.18}$$

Here the tilde upon $F_0(\xi_\parallel)$ denotes a solution of the homogeneous differential equation. The operator $\hat{D}$ has two linearly independent eigenfunctions which are given by

$$\tilde{F}_0^{\pm}(\xi_\parallel) = \exp\left(\pm\frac{1}{\varepsilon_E}\int_0^{\xi_\parallel}\frac{\mathrm{d}\xi'_\parallel}{\Psi_1(\xi'_\parallel)}\right). \tag{2.19}$$

Hence, the solution of the initial inhomogeneous equation (2.15) can be obtained in the ordinary way [5] and it reads

$$F_0(\xi_\parallel) = A_0\tilde{F}_0^-(\xi_\parallel) + B_0\tilde{F}_0^+(\xi_\parallel) - \frac{1}{2\varepsilon_E}\int_0^\infty \begin{vmatrix} \tilde{F}_0^-(\xi_\parallel) & \tilde{F}_0^+(\xi_\parallel) \\ \tilde{F}_0^-(\xi'_\parallel) & \tilde{F}_0^+(\xi'_\parallel) \end{vmatrix} \Phi(\xi'_\parallel)\,\mathrm{d}(\xi'_\parallel) \tag{2.20}$$

where A_0 and B_0 are integration constants, which have to be determined from the boundary conditions to equation (2.15).

For the further transformation of equation (2.20) let us substitute the expression for $\Phi(\xi_\parallel)$ from equation (2.17) into equation (2.20) and change the order of integration over $\xi'_\parallel$ and ω. This yields the following expression:

$$F_0(\xi_\parallel) = A_0\tilde{F}_0^-(\xi_\parallel) + B_0\tilde{F}_0^+(\xi_\parallel) + \int_0^\infty R(\omega,\xi_\parallel)F_0(\omega)\,\mathrm{d}\omega \tag{2.21}$$

where

$$R(\omega,\xi_\parallel) = \frac{1}{2\varepsilon_E}\int_0^{\xi_\parallel} \begin{vmatrix} \tilde{F}_0^-(\xi_\parallel) & \tilde{F}_0^+(\xi_\parallel) \\ \tilde{F}_0^-(\xi'_\parallel) & \tilde{F}_0^+(\xi'_\parallel) \end{vmatrix} K_1(\omega,\xi'_\parallel)\,\mathrm{d}(\xi'_\parallel). \tag{2.22}$$

It is necessary to note that the validity of changing the order of integration in the process of transformation of equation (2.20) into equation (2.21) requires a special mathematical justification [6]. In the general case the justification is a very complicated procedure, especially if it takes into account that we have an unknown function $F_0(\omega)$ under the integral. Here we will assume the validity of this change and later it will be confirmed rigorously.

The result of the transformation which has been performed above consists of the transition from an integro-differential equation (2.10) to a pure integral equation (2.21) with a kernel given by equation (2.22). Equation (2.21) allows us to obtain analytical solutions for $F_0(\xi_\parallel)$ in practically important cases. This will be possibly due to the properties of the function $\Psi_1(\xi_\parallel)$. Before that, let us discuss the boundary conditions for equations (2.8) and (2.21).

First of all, the distribution function $F_0(\xi_\parallel)$ has to be normalized; that is

$$n_0 = 2T_0^{1/2}\int_0^\infty N(\xi_\parallel)F_0(\xi_\parallel)\,\mathrm{d}\xi_\parallel \tag{2.23}$$

where n_0 is the linear electron density in a 1D QWI and $N(\xi_\parallel)$ is the electron density of states. The second condition is the absence of electrons in the range with infinitely high energies, namely

$$F_0(\xi_\parallel) \to 0 \qquad \text{if } \xi_\parallel \to \infty. \tag{2.24}$$

By using the boundary condition from equation (2.24), we obtain the following expression for constant B_0 (for details of the calculation see appendix A):

$$B_0 = \frac{1}{2\varepsilon_E}\int_0^\infty\int_0^\infty K_1(\omega,\xi'_\parallel)\tilde{F}_0^-(\xi'_\parallel)F_0(\omega)\,\mathrm{d}\xi'_\parallel\,\mathrm{d}\omega. \tag{2.25}$$

By substituting B_0 back into equation (2.21) and taking into account equations (2.19) and (2.22) we obtain

$$F_0(\xi_\parallel) = A_0 \exp\left(-\frac{1}{\varepsilon_E}\int_0^{\xi_\parallel} \frac{\mathrm{d}\xi'_\parallel}{\Psi_1(\xi'_\parallel)}\right) + \int_0^\infty H(\omega, \xi_\parallel) F_0(\omega)\,\mathrm{d}\omega. \tag{2.26}$$

Here

$$H(\omega, \xi_\parallel) = \frac{1}{2\varepsilon_E}[H_1(\omega, \xi_\parallel) + H_2(\omega, \xi_\parallel)] \tag{2.27}$$

$$H_1(\omega, \xi_\parallel) = \int_0^{\xi_\parallel} K_1(\omega, \xi'_\parallel) \exp\left(-\frac{1}{\varepsilon_E}\int_{\xi'_\parallel}^{\xi_\parallel} \frac{\mathrm{d}\xi''_\parallel}{\Psi_1(\xi''_\parallel)}\right) \mathrm{d}\xi'_\parallel \tag{2.28}$$

$$H_2(\omega, \xi_\parallel) = \int_{\xi_\parallel}^\infty K_1(\omega, \xi'_\parallel) \exp\left(-\frac{1}{\varepsilon_E}\int_{\xi_\parallel}^{\xi'_\parallel} \frac{\mathrm{d}\xi''_\parallel}{\Psi_1(\xi''_\parallel)}\right) \mathrm{d}\xi'_\parallel. \tag{2.29}$$

the constant A_0 can be calculated from the normalization condition in equation (2.23).

Hence, we have derived the set of equations (2.26)–(2.29) which is very convenient for the analytical analysis in some special but practically important cases. In the general case these equations open up a straightforward way to apply numerical procedures because the integral equation (2.26) is comparably simpler in this respect than is the initial integro-differential equation (2.28). There are many well-developed special numerical methods for the solution of integral equations. Here we restrict our investigation to the analytical solvable cases only.

3. The case of warm electrons under inelastic scattering

The case of warm electrons implies that the electron system is very close to the thermodynamic equilibrium state. That is, the external electric field is sufficiently small that

$$\varepsilon_E^2 \ll 1. \tag{3.1}$$

From the mathematical standpoint the case requires more in-depth analysis. The point is that the small parameter ε_E is a factor at a derivative of a higher order in the integro-differential equation (2.8). If we put $\varepsilon_E = 0$ then we immediately change *the kind* of the equation: the integro-differential equation (2.8) is transformed to the pure integral equation

$$F_0^{(0)}(\xi_\parallel) = \Psi_1(\xi_\parallel)\int_0^\infty K_1(\omega, \xi) F_0^{(0)}(\omega)\,\mathrm{d}\omega \tag{3.2}$$

where the upper subscript '(0)' denotes the solution for the particular case $\varepsilon_E = 0$.

In the general case the solution $F_0(\xi_\parallel)$ of the initial equation does not necessarily tend to $F_0^{(0)}(\xi_\parallel)$ when $\varepsilon_E \to 0$. Detailed investigation has to be performed to solve the problem. Equations of such a kind are singularly perturbed equations [7]. In solving these equations in the general case it is necessary to apply some special methods. For the integro-differential equations such methods were developed in [8, 9].

In accordance with [8] let us transform our initial equation (2.15) into the canonical form

$$\frac{\mathrm{d}Z(\xi_\parallel)}{\mathrm{d}\xi_\parallel} = \frac{1}{\varepsilon_E^2}\left(-\varepsilon_E^2 \frac{\mathrm{d}\ln\Psi_1(\xi_\parallel)}{\mathrm{d}\xi_\parallel} Z(\xi_\parallel) + \frac{1}{\Psi_1^2(\xi_\parallel)}(F_0(\xi_\parallel) + \Psi_1(\xi_\parallel)\Phi(\xi_\parallel))\right)$$
$$\equiv \Lambda[Z, F_0(\xi_\parallel), \varepsilon_E] \tag{3.3}$$

$$\frac{\mathrm{d}F_0(\xi_\parallel)}{\mathrm{d}\xi_\parallel} = Z(\xi_\parallel). \tag{3.4}$$

One can show that the functional $\Lambda[Z, F_0(\xi_\parallel), \varepsilon_E]$ is a continuous function of the parameter ε_E at the special point $\varepsilon_E = 0$. Indeed, if $\varepsilon_E \to 0$ then the numerator in equation (3.3) tends to zero together with the denominator as ε_E^2 (as will be shown later). The numerator tends to $\Psi_1^{-2}(\xi_\parallel)[F_0(\xi_\parallel) + \Psi_1(\xi_\parallel)\Phi(\xi_\parallel)]$, where the expression in the square brackets coincides with equation (3.2) and is equal to zero at $F_0(\xi_\parallel) = F_0^{(0)}(\xi_\parallel)$. The absence of singularity of $\lambda[Z, F_0(\xi_\parallel), \varepsilon_E]$ at the special point $\varepsilon_E = 0$ means that the initial equation (2.15) is an equation of the strictly particular type which is not singularly perturbed despite the presence of a small parameter ε_E at a higher derivative in the equation. Both the boundary conditions from equations (2.23) and (2.24) are in agreement with the solutions of equation (2.15) at $\varepsilon_E = 0$, namely of equation (3.2) as well. The latter circumstance is extremely important for the absence of the singularity of $\Lambda[Z, F_0(\xi_\parallel), \varepsilon_E]$. This is because the solution of equation (3.2) contains only one multiplying constant and only one boundary condition is required for its determination. However, if $\varepsilon_E \neq 0$ then equation (2.15) includes derivatives and two boundary conditions are required for the solution. What this means in the general case is the possibility of a very strong perturbation of the solution of equation (2.15) if $\varepsilon_E \to 0$ and this solution taken at $\varepsilon_E = 0$ will not coincide with the solution of equation (3.2). However, in our strictly particular case the situation is completely different.

By using equation (2.6) we can find the solution of equation (3.2)

$$F_0^{(0)}(\xi_\parallel) = \text{constant} \times \exp(-\xi_\parallel). \tag{3.5}$$

This is the Maxwellian distribution as it must be for the thermodynamic equilibrium state. The multiplying constant is determined by the normalization condition from equation (2.23). As can be seen from equation (3.5), the function $F_0^{(0)}(\xi_\parallel)$ obeys the second boundary condition from equation (2.24) as well.

Thus, equation (2.10) with boundary conditions from equations (2.23) and (2.24) represents the regular rather than the singularly perturbed integro-differential equation. For its solution we can apply ordinary perturbation methods [7]. Another way of looking at it is to use equation (2.26). For reasons which will be substantiated later we will proceed from equation (2.26).

Let us introduce the function

$$J(\xi_\parallel) = \int_0^{\xi_\parallel} \frac{\mathrm{d}\xi'_\parallel}{\Psi_1(\xi'_\parallel)}. \tag{3.6}$$

The first term in equation (2.26) will be negligibly small for all electron energies which satisfy the condition

$$J(\xi_\parallel) > \varepsilon_E. \tag{3.7}$$

Figure 1 shows the dependence of the borderline electron energy $\xi_\parallel^0$ on the dimensionless electric field ε_E in accordance with the equation $J(\xi_\parallel^0) = \varepsilon_E$. At a given ε_E the condition from equation (3.7) is satisfied if $\xi_\parallel > \xi_\parallel^0$. In the range of electric fields which is defined by equation (3.1) we have $\xi_\parallel^0 \ll 1$ and equation (3.7) holds for all actual electron energies.

Under these conditions we can neglect the first term in equation (2.26) and the equation resulting from this is the homogeneous integral equation

$$F_0(\xi_\parallel) = \int_0^\infty H(\omega, \xi_\parallel) F_0(\omega)\, \mathrm{d}\omega. \tag{3.8}$$

First of all note that equation (3.8) is not an equation of Fredholm's type [6]. This is because the kernel $H(\omega, \xi_\parallel)$ does not satisfy the following condition [6]:

$$\int_0^\infty \int_0^\infty H^2(\omega, \xi_\parallel)\, \mathrm{d}\omega\, \mathrm{d}\xi_\parallel < +\infty. \tag{3.9}$$

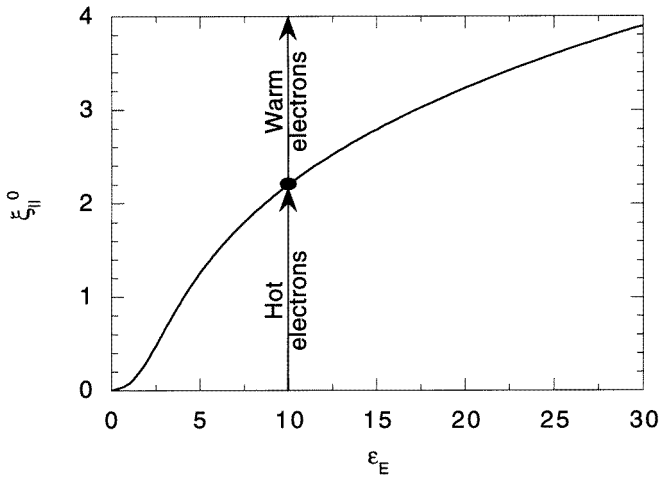

Figure 1. The solution of the equation $J(\xi_{\|}^0) = \varepsilon_E$. The borderline $\xi_{\|}^0(\varepsilon_E)$ defines the electron energy region with hot and warm electrons for a fixed electric field.

In fact, the integral over ω diverges logarithmically at the lower limit point $\omega = 0$, since $H^2(\omega, \xi_{\|}) \propto 1/\omega$ if $\omega \to 0$. This means that we cannot use the ordinary methods [6] for the solution of the integral equation (3.8) and it is necessary to develop a suitable method.

Within the condition from equations (3.1) and (3.7), let us perform the transformation of the kernel $H(\omega, \xi_{\|})$. As seen from equations (2.28) and (2.29), we have under the integrals the products of a slowly varying function of $\xi_{\|}'$, $K_1(\omega, \xi_{\|}')$, and a very rapidly varying function of $\xi_{\|}'$, $\exp[\pm(J(\xi_{\|}) - J(\xi_{\|}'))/\varepsilon_E]$. This exponential function has its maximum value of unity at $\xi_{\|}' = \xi_{\|}$ and very rapidly decreases to e^{-1}, due to condition (3.7), as $\xi_{\|}'$ varies over the narrow interval

$$\Delta\xi_{\|}' = |\xi_{\|} - \xi_{\|}'| \simeq \varepsilon_E \left(\frac{dJ(\xi_{\|})}{d\xi_{\|}}\right)^{-1} \simeq \xi_{\|} \frac{\varepsilon_E}{J(\xi_{\|})} \ll \xi_{\|}. \tag{3.10}$$

Consequently, the main contribution to the integral in equations (2.28) and (2.29) comes from the range $\xi_{\|}'$ which is very close to $\xi_{\|}$, that is, to the upper limit of the integration for equation (2.28) and to the lower one for equation (2.29). This allows us to apply a special Laplace method [10] for the calculations of these integrals.

We have restricted our calculations of $H_{1,2}(\omega, \xi_{\|})$ to the accuracy which is proportional to the third order of the small parameter ε_E. With this approximation equations (2.28) and (2.29) can be written as (for the details of the calculations see appendix B)

$$H_i(\omega, \xi_{\|}) = \varepsilon_E \Psi_1(\xi_{\|}) \Bigg[K_1(\omega, \xi_{\|}) + (-1)^i \varepsilon_E \frac{d}{d\xi_{\|}} (\Psi_1(\xi_{\|}) K_1(\omega, \xi_{\|})) + \varepsilon_E^2 \frac{d}{d\xi_{\|}} \left(\Psi_1(\xi_{\|}) \frac{d}{d\xi_{\|}} (\Psi_1(\xi_{\|}) K_1(\omega, \xi_{\|})) \right) \Bigg] \qquad i = 1, 2. \tag{3.11}$$

As a result we obtain the following expression for the kernel $H(\omega, \xi_{\|})$ in equation (2.27):

$$H(\omega, \xi_{\|}) = \Psi_1(\xi_{\|}) K_1(\omega, \xi_{\|}) + \varepsilon_E^2 \Psi_1(\xi_{\|}) \frac{d}{d\xi_{\|}} \left(\Psi_1(\xi_{\|}) \frac{d}{d\xi_{\|}} (\Psi_1(\xi_{\|}) K_1(\omega, \xi_{\|})) \right). \tag{3.12}$$

Equation (3.12) is combined with equation (3.8) to produce the final equation for $F_0(\xi_\parallel)$ under the condition (3.1). This is

$$F_0(\xi_\parallel) = \Psi_1(\xi_\parallel)\int_0^\infty K_1(\omega,\xi_\parallel)F_0(\omega)\,\mathrm{d}\omega + \varepsilon_E^2\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left[\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left(\Psi_1(\xi_\parallel)\int_0^\infty K_1(\omega,\xi_\parallel)F_0(\omega)\,\mathrm{d}\omega\right)\right]. \tag{3.13}$$

Let us apply an iterative procedure with respect to the small parameters $\varepsilon_E \ll 1$ to the solution of equation (3.12) and present the distribution function $F_0(\xi_\parallel)$ in the form

$$F_0(\xi_\parallel) = f^{(0)}(\xi_\parallel) + \varepsilon_E^2 f^{(1)}(\xi_\parallel). \tag{3.14}$$

On substituting equation (3.14) into equation (3.13) and taking into account equations (3.2) and (3.5) we obtain

$$f^{(0)}(\xi_\parallel) = F_0^{(0)}(\xi_\parallel) = \text{constant} \times \exp(-\xi_\parallel) \tag{3.15}$$

$$f^{(1)}(\xi_\parallel) = \Psi_1(\xi)\int_0^\infty K_1(\omega,\xi_\parallel)f^{(1)}(\omega)\,\mathrm{d}\omega + \Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left[\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left(\Psi_1(\xi_\parallel)\int_0^\infty K_1(\omega,\xi_\parallel)f^{(0)}(\omega)\,\mathrm{d}\omega\right)\right]. \tag{3.16}$$

Combining equations (3.15) and (3.16) yields an equation for $f^{(1)}(\xi_\parallel)$

$$f^{(1)}(\xi_\parallel) = \Psi_1(\xi_\parallel)\int_0^\infty K_1(\omega,\xi_\parallel)f^{(1)}(\omega)\,\mathrm{d}\omega + \Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left(\Psi_1(\xi_\parallel)\frac{\mathrm{d}f^{(0)}(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right). \tag{3.17}$$

As is seen, equations (3.15) and (3.17) are in complete agreement with the initial equation (2.10). This fact is important from the mathematical standpoint. The point here is that equations (3.15) and (3.17) were derived from equation (2.26). As we mentioned above in the process of the derivation of equation (2.26) itself, we have adopted a few mathematical assumptions without rigorous justification. For example, we changed the order of integration in equation (2.21) and the integration and limit calculations in equation (A3) (in appendix A). The agreement obtained among equations (3.17), (2.26) and (2.10) is a good justification of the correctness of equation (2.26).

Returning to the inhomogeneous integral equation (3.17), we recall that this is not an equation of Fredholm's type due to violation of the condition (3.9). Nevertheless, the particular form of the energy-dependence of the second term in equation (3.17) allows us, as will be shown later, to apply the method which is usually used to obtain the solution of equations of Fredholm's type. In particular, the Picar method [6] is appropriate here.

In line with the Picar method, let us present the function $f^{(1)}(\xi_\parallel)$ in the form

$$f^{(1)}(\xi_\parallel) = f^{(0)}(\xi_\parallel)\sum_{i=0} g_i(\xi_\parallel) \tag{3.18}$$

where

$$g_n(\xi_\parallel) = \int_0^\infty k(\omega,\xi_\parallel)g_{n-1}(\omega)\,\mathrm{d}\omega \tag{3.19}$$

$$g_0(\xi_\parallel) = \Psi_1(\xi_\parallel)\left(\Psi_1(\xi_\parallel) - \frac{\mathrm{d}\Psi_1(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right) \tag{3.20}$$

$$k(\omega,\xi_\parallel) = K(\omega,\xi_\parallel)\Big/\int_0^\infty K(\omega,\xi)\,\mathrm{d}\omega. \tag{3.21}$$

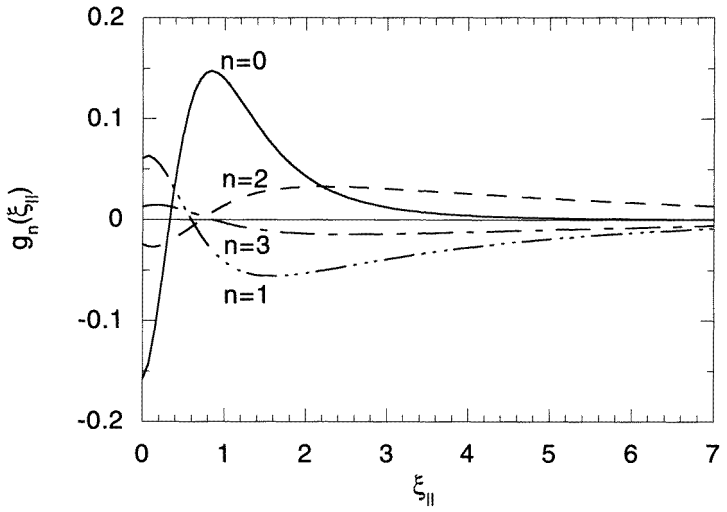

Figure 2. The solution of the integral equation (3.17) in the form of equation (3.18).

The function $k(\omega, \xi_\parallel)$ satisfies the normalization condition

$$\int_0^\infty k(\omega, \xi)\, \mathrm{d}\omega = 1. \tag{3.22}$$

The function $g_0(\xi_\parallel)$ in equation (3.20) is the sign-changing function as seen from equations (2.6) and (2.11) and figure 2. Furthermore, $|g_0(\xi_\parallel)| \lesssim 1$ within the range of the actual energy values. Taking into account the condition (3.22) we find from equation (3.19) that $|g_1(\xi_\parallel)| \ll |g_0(\xi_\parallel)|$ within the same range. Figure 2 confirms this inequality. This is true for all $g_n(\xi_\parallel)$ if $n \geqslant 1$ and this is why we can apply the Picar method for the solution of equation (3.16). As a consequence of these inequalities we need retain in equation (3.18) only the first term for the actual energy range and we obtain

$$f^{(1)}(\xi_\parallel) \sim f^{(0)}(\xi_\parallel) g_0(\xi_\parallel) = f^{(0)}(\xi_\parallel)\Psi_1(\xi_\parallel)\left(\Psi_1(\xi_\parallel) - \frac{\mathrm{d}\Psi_1(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right). \tag{3.23}$$

Combination of equations (3.14), (3.15) and (3.23) gives us the final expression for the distribution function $F_0(\xi_\parallel)$ for the case of warm electrons:

$$F_0(\xi_\parallel) = \text{constant} \times \left[1 + \varepsilon_E^2 \Psi_1(\xi_\parallel)\left(\Psi_1(\xi_\parallel) - \frac{\mathrm{d}\Psi_1(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right)\right] \exp(-\xi_\parallel). \tag{3.24}$$

If we substitute this expression into equations (3.3) and (3.4) we will see that the expression in square brackets in equation (3.3) tends to zero as ε_E^2 if $\varepsilon_E \to 0$, as it was assumed to do in the process of investigation of equations (3.3) and (3.4).

It is relevant to introduce the momentum relaxation time $\tau(\xi_\parallel)$ into equation (3.24). By using equations (2.4), (2.7), (2.9) and (2.11) we obtain

$$F_0(\varepsilon_\parallel) = \text{constant} \times \left[1 + \frac{e^2 E_x^2 \lambda_\varepsilon^2(\varepsilon_\parallel)}{T_0^2}\left(1 - \frac{T_0}{\lambda_\varepsilon(\varepsilon_\parallel)}\frac{\mathrm{d}\lambda_\varepsilon(\varepsilon_\parallel)}{\mathrm{d}\varepsilon_\parallel}\right)\right] \exp\left(-\frac{\varepsilon_\parallel}{T_0}\right) \tag{3.25}$$

where

$$\lambda_\varepsilon(\varepsilon_\parallel) = v(\varepsilon_\parallel)\tau(\varepsilon_\parallel) \tag{3.26}$$

is the free path of the electron with energy $\varepsilon_\parallel$ and velocity $v(\varepsilon_\parallel) = (2\varepsilon_\parallel/m^*)^{1/2}$.

The criterion of validity of equation (3.25) is given by equation (3.1). Taking into account that the actual energy values for the warm-electron case are within the range $\varepsilon_\parallel \gtrsim T_0$ we can write the criterion in the form

$$eE_x\lambda_\varepsilon(T_0) < T_0. \tag{3.27}$$

Equation (3.27) means that the electrons gain little energy from the electric field compared with the equilibrium value. In this case the second term in equation (3.25) is small compared with the first one. The distribution functions in equations (2.3) and (3.25) describe all kinetic properties of warm electrons in a 1D QWI under inelastic scattering by the acoustic phonons.

4. The case of hot electrons under inelastic scattering

In contradistinction to the foregoing section 3, here we suppose that the 1D electron system is far from the thermodynamic equilibrium state. The situation of the hot electrons is realized in a strong electric field when

$$\varepsilon_E^2 > 1. \tag{4.1}$$

We suppose as well that the following condition holds for the actual electron energy values:

$$J(\xi_\parallel) < \varepsilon_E. \tag{4.2}$$

As one can see from figure 1, equations (4.1) and (4.2) define the energy region $\xi_\parallel > \xi_\parallel^0 > 1$. The majority of electrons will be distributed within this region if the electric field is sufficiently strong according to equation (4.1).

Let us consider equation (2.26) for this case. The structure of this equation suggests a main role for the first term within the conditions from equations (4.1) and (4.2). This means that the symmetric distribution function is equal to

$$F_0(\xi_\parallel) = A_0 \exp\left(-\frac{1}{\varepsilon_E}\int_0^{\xi_\parallel} \frac{\mathrm{d}\xi_\parallel'}{\Psi_1(\xi_\parallel')}\right). \tag{4.3}$$

To prove that the function from equation (4.3) is a good approximation for the solution of equation (2.26) it is necessary to estimate the second term in equation (2.26). Substitution of the function $F_0(\xi_\parallel)$ into the second term in equation (2.26) gives the expression

$$\Delta F_0(\xi_\parallel) \equiv \int_0^\infty H(\omega,\xi_\parallel)F_0(\omega)\,\mathrm{d}\omega = A_0\int_0^\infty H(\omega,\xi_\parallel)\exp\left(-\frac{1}{\varepsilon_E}\int_0^\omega \frac{\mathrm{d}\omega'}{\Psi_1(\omega')}\right)\mathrm{d}\omega. \tag{4.4}$$

For the sake of convenience of comparing $F_0(\xi_\parallel)$ and $\Delta F_0(\xi_\parallel)$ we can represent $\Delta F_0(\xi_\parallel)$ in the form

$$\Delta F_0(\xi_\parallel) = F_0(\xi_\parallel)\delta(\xi_\parallel,\varepsilon_E) \tag{4.5}$$

where

$$\delta(\xi_\parallel,\varepsilon_E) = \int_0^\infty H(\omega,\xi_\parallel)\exp\left(-\frac{1}{\varepsilon_E}\int_{\xi_\parallel}^\omega \frac{\mathrm{d}\omega'}{\Psi_1(\omega')}\right)\mathrm{d}\omega. \tag{4.6}$$

By using equations (2.27)–(2.29) we can perform an analytical calculation of $\delta(\xi_\parallel,\varepsilon_E)$. After some cumbersome and routine transformations we obtain the following strong inequality:

$$\delta(\xi,\varepsilon_E) \ll \left(\frac{J(\xi_\parallel)}{\varepsilon_E}\right)^{1/6}\exp\left(\frac{J(\xi_\parallel)}{\varepsilon_E}\right). \tag{4.7}$$

The right-hand side of equation (4.7) for the actual values of the electron energy and electric fields has a value of the order of unity, and consequently $\delta(\xi_\parallel, \varepsilon_E) \ll 1$. By this is meant that $\Delta F_0(\xi_\parallel) \ll F_0(\xi_\parallel)$ and, therefore, equation (4.3) is a good approximation for the solution of equation (2.26). If we turn back to the initial equation (2.8) we can see the approximation equation (4.3) means that we neglect the relaxation processes associated with scattering into the κ_x state in comparison with the scattering out of the κ_x state.

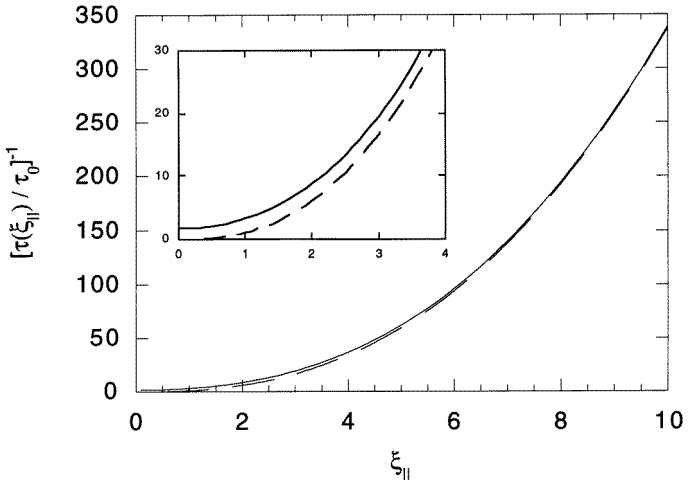

Figure 3. The scattering rate as a function of electron energy for inelastic electron–acoustic-phonon interaction. The full line represents results of equations (2.4)–(2.6); the broken line corresponds to equation (4.14).

Let us find the antisymmetrical distribution function for these conditions. By substituting the symmetrical distribution function $F_0(\xi_\parallel)$ from equation (4.3) into equation (2.3), we obtain after some transformations

$$F^-(\kappa_x) = -\frac{\kappa_x E_x}{|\kappa_x||E_x|} F_0(\xi_\parallel). \tag{4.8}$$

By using equations (4.3) and (4.8) we can calculate the total distribution function $F(\kappa_x)$. We obtain the following expression:

$$F(\kappa_x) = A_0 \left(1 - \frac{\kappa_x E_x}{|\kappa_x||E_x|}\right) \exp\left(-\frac{1}{\varepsilon_E}\int_0^{\xi_\parallel} \frac{\mathrm{d}\xi'_\parallel}{\Psi_1(\xi'_\parallel)}\right). \tag{4.9}$$

It is useful to present this expression in another equivalent form. By making use of the expressions for the electron density state and for the scattering rate from equation (2.4) we obtain

$$F(\kappa_x) = A_0 \left(1 - \frac{\kappa_x E_x}{|\kappa_x||E_x|}\right) \exp\left(-\frac{\pi\hbar}{e|E_x|}\int_0^{\varepsilon_\parallel(\kappa_x)} \frac{N(\varepsilon'_\parallel)\,\mathrm{d}\varepsilon'_\parallel}{\tau(\varepsilon'_\parallel)}\right). \tag{4.10}$$

It follows from equation (4.10) that the energy-dependence of the momentum relaxation time $\tau(\xi_\parallel)$ describes completely the hot electron distribution in a 1D QWI. In our case $\tau(\xi_\parallel)$ is given by equations (2.4)–(2.6). Its energy-dependence is shown in figure 3.

For some limiting values of the energy one can obtain from these equations the following expressions:

$$\frac{1}{\tau(\xi_\parallel)} = \frac{1}{\tau_0}\left(\frac{3\sqrt{\pi}}{4}\zeta(5/2) + \frac{4}{3}\xi_\parallel^{3/2}\right) \qquad \text{if } \xi_\parallel < 1 \tag{4.11}$$

$$\frac{1}{\tau(\xi_\parallel)} = \frac{1}{\tau_0}\left(\frac{2\zeta(3)}{\xi_\parallel^{1/2}} + \frac{16}{15}\xi_\parallel^{5/2}\right) \qquad 1 < \xi_\parallel < \frac{(8m^*s^2W_0)^{1/2}}{T_0} \tag{4.12}$$

where

$$\zeta(q) = \frac{1}{\Gamma(q)}\int_0^\infty \frac{x^{q-1}}{e^x - 1}\,\mathrm{d}x \tag{4.13}$$

is the zeta function of Riemann.

The first term in the square brackets in equations (4.11) and (4.12) corresponds to the absorption and the second one to the emission of the acoustic phonons. For the case of the hot electrons $\tau(\xi_\parallel)$ is given by equation (4.12), in which we can neglect the first term. This means that emission processes give the main contribution to the hot electron momentum relaxation rate:

$$\frac{1}{\tau(\xi_\parallel)} = \frac{16}{15}\tau_0^{-1}\xi_\parallel^{5/2}. \tag{4.14}$$

As one can see from figure 3, this is a good approximation for $\tau^{-1}(\xi_\parallel)$ if $\xi_\parallel > 1$. By using equation (4.14) we obtain the following expression for the distribution function for the discussed limiting case:

$$F(\kappa_x) = A_0\left(1 - \frac{\kappa_x E_x}{|\kappa_x||E_x|}\right)\exp\left(-\frac{16}{45}\frac{\xi_\parallel^3}{\varepsilon_E}\right). \tag{4.15}$$

The expressions for the total distribution function $F(\kappa_x)$ from equations (4.10) and (4.11) show us that $F(\kappa_x)$ is a strongly anisotropic function in momentum space:

$$F(\kappa_x) = \begin{cases} 2F_0(\varepsilon_\parallel) & \text{if } \kappa_x E_x < 0 \\ 0 & \text{if } \kappa_x E_x > 0. \end{cases} \tag{4.16}$$

Physically this means that all electrons move ballistically in κ_x-space along straightforward trajectories in the direction which is opposite to the direction of the electric field (the last is obvious if we take into account the negative sign of the electron charge). In the steady state the electron motion is governed by the momentum balance equation

$$-eE_x\tau(\tilde{\varepsilon}_\parallel) = \hbar\kappa_x(\tilde{\varepsilon}_\parallel). \tag{4.17}$$

The electric field pushes electrons into the high-energy region, where the increase in energy results in an increase in the scattering probability in accordance with equation (4.14). There is some 'balance' value of the electron energy $\tilde{\varepsilon}_\parallel$ which is given by the solution of equation (4.17) and which depends on the electric field E_x. When the electron reaches the energy $\tilde{\varepsilon}_\parallel$ and momentum $\hbar\kappa_x(\tilde{\varepsilon}_\parallel)$ it emits a 3D acoustic phonon with the wavevector $\boldsymbol{q} \equiv (q_x, q_\perp)$ and with the energy $\hbar\omega_{\boldsymbol{q}} = \hbar sq$. In accordance with longitudinal momentum conservation, $q_x = \kappa_x(\tilde{\varepsilon}_\parallel)$; that is, the electron loses its momentum due to interaction with the q_x component of the phonon wavevector $\boldsymbol{q}$. The portion of the phonon energy $\hbar s|q_x| = (2m^*s^2\tilde{\varepsilon}_\parallel)^{1/2}$ which corresponds to this 'longitudinal' interaction is very small compared with $\tilde{\varepsilon}_\parallel$. In accordance with energy conservation and taking into account the relationship $q_\perp \gg |q_x|$, the electron loses its energy due mainly to the interaction with the $\boldsymbol{q}_\perp$ component of $\boldsymbol{q}$, namely due to emission of the acoustic phonon with the energy

$\hbar\omega_{\boldsymbol{q}} \simeq \hbar s q_\perp \sim \tilde{\varepsilon}_\parallel$. Because this is a strongly inelastic interaction the electron loses all its energy and is scattered down to the sub-band bottom. After this, the process, which is governed by equation (4.17), will be repeated periodically again.

The rigorous kinetic description of the process is given by the distribution function from equation (4.10). As one can see, the exponential function includes the momentum balance equation from equation (4.17) with the average over the density of states.

The strongly anisotropic electron distribution corresponds to the so-called electron streaming regime. The latter is more familiar in the context of the inelastic interaction of electrons with optical phonons [11–13]. A regime similar to that discussed here was investigated in [14] for 1D electrons in a QWI interacting with bulk acoustic phonons by using the Monte Carlo technique. Our analytical results coincide qualitatively with those from the Monte Carlo analysis, but there are some quantitive distinctions for the electric-field-dependence of the kinetic coefficients of a 1D hot-electron gas. We will explain the origin of the discrepancies later in the corresponding section. The new strongly anisotropic distribution function given by equation (4.10) for the 1D hot electrons under inelastic scattering by the acoustic phonons is the main result of this section.

5. The case of hot electrons: the zero-point lattice and quasielasticity

With increasing electric field E_x the electrons penetrate into the high-energy region where equation (4.17) has no solution. The inelastic condition for the electron–acoustic-phonon interaction is broken. It follows from the previous analysis of papers I and II that, if the electron energy is within the range

$$(8m^*s^2W_0)^{1/2} < \varepsilon_\parallel(\kappa_x) < \tfrac{3}{2}W_0 \tag{5.1}$$

then the electrons occupy the first sub-band only and the electron–acoustic-phonon interaction is a quasi-elastic one.

By assuming the same as above for the low-lattice-temperature case (see equation (2.2)), we can use in the Boltzmann equations (equations (2.11) and (2.12) in paper II) the expressions for the antisymmetrical operator $\hat{I}F^-(\kappa_x)$ (equations (3.4) and (3.5) in paper II) and for the symmetrical operator $\hat{I}F_0(\varepsilon_\parallel)$ (equation (4.19) in paper II), where it is necessary to put $\nu' = \nu = 1$. As a result the antisymmetrical distribution $F^-(\kappa_x)$ is given by equation (2.3), where the momentum relaxation time $\tau(\varepsilon_\parallel)$ is equal to

$$\frac{1}{\tau(\varepsilon_\parallel)} = w_0 L_x N(\varepsilon_\parallel) \sum_{\boldsymbol{q}_\perp} G_{11}^2(\boldsymbol{q}_\perp) \left(q_\perp^2 + \frac{8m^*\varepsilon_\parallel}{\hbar^2} \right)^{1/2}. \tag{5.2}$$

To obtain equation (5.2) we have used the zero-point-lattice approximation, $N_{\boldsymbol{q}} \simeq 0$, and have performed the summation over q_x. The summation over $\boldsymbol{q}_\perp$ cannot be performed analytically in equation (5.2) with the form factor $G_{11}^2(\boldsymbol{q}_\perp)$ for a rectangular QWI, but a qualitative picture of the energy-dependence of the momentum relaxation rate can be given as follows.

If the electron energy is within the range

$$(8m^*s^2W_0)^{1/2} < \varepsilon_\parallel < W_0 \tag{5.3}$$

we can neglect the second term in the square brackets and obtain

$$\frac{1}{\tau(\varepsilon_\parallel)} \simeq w_0 L_x N(\varepsilon_\parallel) \sum_{\boldsymbol{q}_\perp} G_{11}^2(\boldsymbol{q}_\perp) \boldsymbol{q}_\perp \simeq \varepsilon_\parallel^{-1/2}. \tag{5.4}$$

If the electron energy is within the range

$$W_0 < \varepsilon_\parallel < \tfrac{3}{2}W_0 \tag{5.5}$$

we can neglect the first term in equation (5.2), and obtain

$$\frac{1}{\tau(\varepsilon_\parallel)} \simeq w_0 L_x \left(\frac{8m^*}{\hbar^2}\right)^{1/2} N(\varepsilon_\parallel)\varepsilon_\parallel^{1/2} \sum_{\boldsymbol{q}_\perp} G_{11}^2(\boldsymbol{q}_\perp) \simeq \varepsilon_\parallel^0. \tag{5.6}$$

If we combine equations (4.11), (4.12), (5.4) and (5.6) together, we will see that the scattering rate has a plateau in the low-energy region, then it increases with energy as $\varepsilon_\parallel^{5/2}$, passes through a maximum at energy $\varepsilon_\parallel \simeq (8m^*s^2W_0)^{1/2}$, decreases with energy as $\varepsilon_\parallel^{-1/2}$ up to the energy $\varepsilon_\parallel \simeq W_0$ and saturates in the high-energy region $\varepsilon_\parallel > W_0$. The saturation of the scattering rate in the high-energy region is a very important feature of the electron kinetics in a 1D QWI under zero-point-lattice conditions. As we will see later, it results in the stabilization of the electron distribution and it prevents the runaway effect for the electrons due to interaction with the acoustic phonons only.

The symmetrical distribution function $F_0(\varepsilon_\parallel)$ is governed by the Boltzmann equation which has the Fokker–Planck form

$$-\frac{e^2E_x^2}{n\hbar}\frac{1}{N(\varepsilon_\parallel)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon_\parallel}\left(v(\varepsilon_\parallel)\tau(\varepsilon_\parallel)\frac{\mathrm{d}F_0(\varepsilon_\parallel)}{\mathrm{d}\varepsilon_\parallel}\right) = \frac{4m^*s}{\hbar}w_0L_x\frac{1}{N(\varepsilon_\parallel)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon_\parallel}\left[N^2(\varepsilon_\parallel)\left(A(\varepsilon_\parallel)F_0(\varepsilon_\parallel) + C(\varepsilon_\parallel)\frac{\mathrm{d}F_0(\varepsilon_\parallel)}{\mathrm{d}\varepsilon_\parallel}\right)\right] \tag{5.7}$$

where $v(\varepsilon_\parallel) = (2\varepsilon_\parallel/m^*)^{1/2}$, and the coefficients $A(\varepsilon_\parallel)$ and $C(\varepsilon_\parallel)$ are as given in paper II (see equations (4.10) and (4.20) in II, respectively).

In accordance with our general approach we will obtain the solution of equation (5.7) for an arbitrary shape of the form factor $G_{11}^2(\boldsymbol{q}_\perp)$ which determines the energy-dependence of the coefficients $A(\varepsilon_\parallel)$, $C(\varepsilon_\parallel)$ and the momentum relaxation time $\tau(\varepsilon_\parallel)$. The solution of equation (5.7) is

$$F_0(\varepsilon_\parallel) = B_0 \exp\left(-\int_0^{\varepsilon_\parallel} \frac{A(\varepsilon'_\parallel)\,\mathrm{d}\varepsilon'_\parallel}{\dfrac{\hbar\bar{q}(\varepsilon'_\parallel)}{4m^*s}[eE_x v(\varepsilon'_\parallel)\tau(\varepsilon'_\parallel)]^2 + C(\varepsilon'_\parallel)}\right) \tag{5.8}$$

where B_0 is the normalization constant and $\bar{q}(\varepsilon_\parallel)$ is defined by the relation

$$\bar{q}(\varepsilon_\parallel) = \sum_{\boldsymbol{q}_\perp} G_{11}^2(\boldsymbol{q}_\perp)\left(q_\perp^2 + \frac{8m^*\varepsilon_\parallel}{\hbar^2}\right)^{1/2}. \tag{5.9}$$

Let us investigate the asymptotic behaviour of the distribution function $F_0(\varepsilon_\parallel)$ at large values of $\varepsilon_\parallel$. We have that $A(\varepsilon_\parallel) \sim \varepsilon_\parallel$, $C(\varepsilon_\parallel) \sim \varepsilon_\parallel^{3/2}$, $\tau(\varepsilon_\parallel) \sim \varepsilon^0$ and $\bar{q}(\varepsilon_\parallel) \sim \varepsilon_\parallel^{1/2}$. As a result we obtain for $F_0(\varepsilon_\parallel)$

$$F_0(\varepsilon_\parallel) \sim B_0 \exp\left(-\frac{\varepsilon_\parallel^{1/2}}{E_x^2}\right). \tag{5.10}$$

This distribution function decreases quite rapidly when $\varepsilon_\parallel$ increases. The normalization integral in equation (2.23) with $F_0(\varepsilon_\parallel)$ given by equation (5.10) has a finite value. It means that no runaway effect for the electrons occurs and $F_0(\varepsilon_\parallel)$ in equation (5.8) is a stationary solution of the Boltzmann equation for 1D electrons in a QWI interacting with the acoustic phonons under zero-point-lattice conditions. We will see later that this does not hold for the equipartition approximation and electrons do run away.

The physical reason why the electron distribution is stabilized under the conditions discussed above is as follows. With increasing electron energy $\varepsilon_\parallel$ the momentum scattering rate decreases in accordance with equation (5.2) due to the decreasing electron density of states. However, following the increase in $\varepsilon_\parallel$, the characteristic value of the phonon wavevector $\bar{q}(\varepsilon_\parallel)$ which participates in the interaction with electrons also increases. By using equation (5.9) we can write equation (5.2) in the form

$$\frac{1}{\tau(\varepsilon_\parallel)} = w_0 L_x N(\varepsilon_\parallel)\bar{q}(\varepsilon_\parallel). \tag{5.11}$$

Since $N(\varepsilon_\parallel) \sim \varepsilon_\parallel^{-1/2}$ and $\bar{q}(\varepsilon_\parallel) \sim \varepsilon_\parallel^{1/2}$, the decrease in the density of states is compensated by the increase in the characteristic phonon wavevector emitted by the electrons. Note that here $|q_x| > q_\perp$ and electrons emit phonons with wavevectors along the QWI axis. The momentum scattering rate saturates and gives the distribution function equation (5.10).

It is necessary to point out that the range of electric fields E_x within which the distribution $F_0(\varepsilon_\parallel)$ in equation (5.8) holds is not very large. In accordance with equation (5.10) electrons increase their energy very rapidly with increasing electric field. As a result they will populate the second sub-band as well and it must be taken into account. Here we restrict our investigation to the first-sub-band approximation only, but as it is obvious from a previous analysis that any occupation of the upper sub-bands can only improve the situation and prevent the runaway effect for the electrons.

6. Solution of the Boltzmann equation based on equipartition

Let us now investigate the case of high lattice temperature. As was shown in the previous papers I and II, if the lattice temperature is within the range $T_0 > (8m^*s^2W_0)^{1/2}$, then both the equipartition approximation for the acoustic phonons and the quasi-elastic approximation for the electron–acoustic-phonon interaction are realized if the electron energy is within the range $\varepsilon - W_\nu > (8m^*s^2W_0)^{1/2}$.

Accordingly, we will study here the non-equilibrium kinetics of 1D electrons under the conditions defined by

$$(8m^*s^2W_0)^{1/2} < \varepsilon - W_\nu < \frac{T_0^2}{8m^*s^2} \tag{6.1}$$

$$T_0 > (8m^*s^2W_0)^{1/2}. \tag{6.2}$$

Equation (6.2) and the left-hand side of equation (6.1) show us that the electron–phonon interaction is quasi-elastic for the majority of electrons even under thermodynamic equilibrium conditions. Hence, we can ignore the inelastic contribution to the total relaxation process because this contribution comes from the small energy region within which $\varepsilon - W_\nu < (8m^*s^2W_0)^{1/2} < T_0$.

Under these conditions the antisymmetrical collision operator $\hat{I}F_\nu^-(\kappa_x)$ is given by equations (3.4) and (3.6) in paper II, and the symmetric collision operator $\hat{I}F_{0\nu}(\varepsilon)$ is given by equations (4.9)–(4.11) in paper II. As will be shown later, it is possible to obtain an analytical solution of the Boltzmann equation taking into account the occupation of many sub-bands and this is why we will use here the total electron energy ε rather than the kinetic energy $\varepsilon_\parallel$. For the rectangular QWI this problem was solved in [15] and here we will follow the method proposed in [15].

The solution of the Boltzmann equation for the antisymmetrical function $F_\nu^-(\kappa_x)$ is

$$F_\nu^-(\kappa_x) = \frac{1}{\hbar} e E_x \tau_\nu(\varepsilon) \frac{\mathrm{d}F_{0\nu}(\varepsilon)}{\mathrm{d}\kappa_x} \tag{6.3}$$

where the momentum relaxation time $\tau_\nu(\varepsilon)$ is equal to

$$\frac{1}{\tau_\nu(\varepsilon)} = \sum_{\nu'} \frac{1}{\tau_{\nu\nu'}(\varepsilon)} \tag{6.4}$$

$$\frac{1}{\tau_{\nu\nu'}(\varepsilon)} = 2w_0 \frac{T_0}{\hbar s} L_x N_{\nu'}(\varepsilon) \sum_{\boldsymbol{q}_\perp} G^2_{\nu\nu'}(\boldsymbol{q}_\perp). \tag{6.5}$$

We have put $\varepsilon_\nu(\kappa_x) = \varepsilon$ in equations (6.3)–(6.5).

By substituting $F^-_\nu(\kappa_x)$ from equation (6.3) into the Boltzmann equation for the symmetrical distribution function and averaging this equation over the points of constant energy, as was done in equation (4.1) in paper II, we obtain the Boltzmann equation for the symmetrical distribution function $F_{0\nu}(\varepsilon)$

$$\begin{aligned} -\frac{e^2E_x^2}{\pi\hbar}\frac{1}{N(\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(v_\nu(\varepsilon)\tau_\nu(\varepsilon)\frac{\mathrm{d}F_{0\nu}(\varepsilon)}{\mathrm{d}\varepsilon}\right) \\ = \frac{4m^*s}{\hbar} w_0 L_x \sum_{\nu'} \frac{1}{N_{\nu'}(\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[N^2_{\nu'}(\varepsilon)A_{\nu\nu'}(\varepsilon)\left(F_{0\nu'}(\varepsilon) + T_0\frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon}\right)\right] \\ + \sum_{\nu'} \frac{F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon)}{\tau_{\nu\nu'}(\varepsilon)} \end{aligned} \tag{6.6}$$

where we have introduced the notation

$$v_\nu(\varepsilon) = \left(\frac{2}{m^*}(\varepsilon - W_\nu)\right)^{1/2}. \tag{6.7}$$

From the mathematical point of view equation (6.6) is a system of second-order differential equations. The number of equations is equal to the number of occupied sub-bands.

To solve equation (6.6) we will use the fact that the evaluation of the last term in equation (6.6) and the penultimate one is approximately equal to

$$F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon) \simeq \frac{2m^*s^2}{T_0} F_{0\nu}(\varepsilon). \tag{6.8}$$

Physically this means that the function distribution is changed under inter-sub-band electron scattering even in the elastic approximation. The last term in equation (6.6) describes this process. The penultimate term in equation (6.6) describes the relaxation of the function distribution due to the quasi-elastic electron–acoustic-phonon interaction and it is obvious that this term is proportional to the small parameter of quasi-elasticity $\delta = 2m^*s^2/T_0 \ll 1$.

Hence, we can apply perturbation theory with respect to the parameter δ to solve equation (6.6). The zeroth-order equation is

$$\sum_{\nu'} \frac{F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon)}{\tau_{\nu\nu'}(\varepsilon)} = 0. \tag{6.9}$$

Its solution gives us the equi-occupation condition

$$F_{0\nu'} = F_{0\nu}(\varepsilon) = F_0(\varepsilon) \tag{6.10}$$

that is, the distribution function of the electrons with the energy ε is the same for all sub-bands and does not depend on the sub-band index ν.

Substitution of equation (6.10) into equation (6.6) gives us the first-order equation for $F_0(\varepsilon)$. To transform this equation into the form of a continuity equation in an energy space

of the Fokker–Planck type, let us multiply the equation by the density of states $N_\nu(\varepsilon)$ and sum over ν. We obtain

$$-\frac{e^2E_x^2}{\pi\hbar}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\sum_\nu v_\nu(\varepsilon)\tau_\nu(\varepsilon)\frac{\mathrm{d}F_0(\varepsilon)}{\mathrm{d}\varepsilon}\right) = \frac{4m^*s}{\hbar}w_0L_x\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\sum_{\nu\nu'}N_\nu(\varepsilon)N_{\nu'}(\varepsilon)A_{\nu\nu'}(\varepsilon)\left(F_0(\varepsilon)+T_0\frac{\mathrm{d}F_0(\varepsilon)}{\mathrm{d}\varepsilon}\right)\right]. \tag{6.11}$$

In the process of transformation of the right-hand side of equation (6.11) we have used the following relationship:

$$\sum_{\nu\nu'}\frac{N_\nu(\varepsilon)}{N_{\nu'}(\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}[N_{\nu'}^2(\varepsilon)R_{\nu\nu'}(\varepsilon)] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\sum_{\nu\nu'}N_\nu(\varepsilon)N_{\nu'}(\varepsilon)R_{\nu\nu'}(\varepsilon) \tag{6.12}$$

where $R_{\nu\nu'}(\varepsilon)$ is an arbitrary function obeying the property

$$R_{\nu\nu'}(\varepsilon) = R_{\nu'\nu}(\varepsilon). \tag{6.13}$$

In our case we have

$$R_{\nu\nu'}(\varepsilon) = A_{\nu\nu'}(\varepsilon)\left(F_0(\varepsilon)+T_0\frac{\mathrm{d}F_0(\varepsilon)}{\mathrm{d}\varepsilon}\right) \tag{6.14}$$

and that this function satisfies equation (6.13). The solution of equation (6.11) is

$$F_0(\varepsilon) = C_0\exp\left[-\frac{1}{T_0}\int^\varepsilon\left(1+\frac{e^2E_x^2}{4\pi m^*sw_0L_xT_0}\frac{\sum_\nu v_\nu(\varepsilon')\tau_\nu(\varepsilon')}{\sum_{\nu\nu'}(\varepsilon')N_{\nu'}(\varepsilon')A_{\nu\nu'}(\varepsilon')}\right)^{-1}\mathrm{d}\varepsilon'\right] \tag{6.15}$$

where C_0 is the normalization constant.

For a rectangular 1D QWI the coefficients $A_{\nu\nu'}(\varepsilon)$ are given by equation (4.14) in paper II and the distribution function $F_0(\varepsilon)$ can be presented in the form

$$\begin{aligned}F_0(\varepsilon) = C_0\exp\Bigg[&-\frac{1}{T_0}\int^\varepsilon\mathrm{d}\varepsilon'\Bigg(1+\frac{(\pi eE_x\lambda_a)^2}{4m^*s^2W_\perp}\\&\times\left[\sum_\nu\left(\sum_{\nu'}\gamma_\nu(\varepsilon)\gamma_{\nu'}(\varepsilon)W_\perp B_{\nu\nu'}\right)^{-1}\right]\\&\times\Bigg[2\sum_{\nu\nu'}\gamma_\nu(\varepsilon)\gamma_{\nu'}(\varepsilon)\varepsilon+\sum_\nu\Bigg(\sum_{p'}\gamma_{np'}(\varepsilon)(\varepsilon-n^2W_{0y})\\&+\sum_{n'}\gamma_{n'p}(\varepsilon)(\varepsilon-p^2W_{0z})\Bigg)\gamma_\nu(\varepsilon)+\frac{1}{2}\sum_\nu 1\Bigg]^{-1}\Bigg)^{-1}\Bigg]\end{aligned} \tag{6.16}$$

where we have introduced the following notations: $W_\perp = (W_{0y}W_{0z})^{1/2}$, $\gamma_\nu(\varepsilon) = (\varepsilon - W_\nu)^{-1/2}$ and λ_a is the electron free path in a bulk semiconductor material for the acoustic-phonon scattering [16]

$$\lambda_a = \frac{\pi\hbar^4\rho s^2}{\Xi_a^2(m^*)^2T_0}. \tag{6.17}$$

The coefficient $B_{\nu\nu'}$ is given by equation (4.15) in paper II.

The distribution function in equations (6.15) and (6.16) describes all the kinetic properties of 1D electrons in a QWI interacting with 3D acoustic phonons under equipartition conditions. For a rectangular QWI this problem was originally solved in [15]. However, comparison of our expression for $F_0(\varepsilon)$ in equation (6.6) with the analogous expression

in [15] (equation (18)) shows us that there are some differences between these expressions and this is why we present here $F_0(\varepsilon)$ for this particular case. Unfortunately, the authors of [15] did not give the Boltzmann equation for $F_0(\varepsilon)$ in a 1D electron gas of a QWI and it is difficult to see the origin of the discrepancies between our results.

In the particular case in which the electrons occupy only the first sub-band ($\nu' = \nu = 1$) in a QWI of square cross section ($L_y = L_z \equiv L_\perp$, $W_\perp \equiv W_\perp^0 = \pi^2\hbar^2/(2m^*L_\perp^2)$) we obtain from equation (6.6)

$$F_0(\varepsilon) = C_0 \exp\left[-\frac{1}{T_0}\int^{\varepsilon} \mathrm{d}\varepsilon' \left(1 + \varepsilon_E'^2 \frac{(\varepsilon - 2W_\perp^0)^2}{(3\varepsilon - 2W_\perp)W_\perp^0}\right)^{-1}\right]. \tag{6.18}$$

Here we have introduced the dimensionless electric field

$$\varepsilon_E'^2 = \frac{2\pi^2}{27}\frac{(eE_x\lambda_a)^2}{m^*s^2W_\perp^0} \equiv \frac{E_x^2}{E_0^2} \tag{6.19}$$

where

$$E_0 = \frac{3\sqrt{3}}{\pi\sqrt{2}}\frac{(m^*s^2w_\perp^0)^{1/2}}{e\lambda_a}. \tag{6.20}$$

We can see that, even in this limiting case, there is an additional factor $\pi^2/18$ which decreases the characteristic electric field compared with the similar expression in [15] (equation (19)).

In the high-energy region, where $\varepsilon > 2W_\perp^0$, the asymptotic behaviour of $F_0(\varepsilon)$ in equation (6.18) is given by

$$F_0(\varepsilon) \simeq C_0 \exp\left[-\frac{W_\perp^0}{T_0}\left(\frac{9}{\pi\varepsilon_E'}\right)^2 \ln\left(\frac{\varepsilon}{W_\perp^0}\right)\right]. \tag{6.21}$$

This distribution function decreases very slowly when ε increases, resulting in the divergence of the normalization integral in equation (2.23). This means that a runaway effect occurs for hot 1D electrons in a QWI under these conditions. To stabilize the electron system and to obtain a steady state electron distribution it is necessary to take into account besides the acoustic phonons another mechanism of interaction, for example, with the optical phonons. Another possibility is to take into account a transition of the electrons to the classical range of energies [15], within which the size of electronic quantization is suppressed and electrons obey the Davydov–Druyvestyn [17] distribution function.

7. The macroscopic characteristics of a non-equilibrium 1D electron gas

The distribution functions derived in previous sections give us a comprehensive kinetic description of the 1D electron gas in a QWI. In the general case the calculation of the kinetic coefficients can be performed via numerical integration of the corresponding expressions containing these functions.

It is noteworthy that the shape of the quantizing potential is included in the expressions for the distribution functions in an arbitrary form. The shape of the quantum well defines the electron energy spectrum in a QWI and the form factor for the electron–phonon interaction. To calculate the kinetic coefficients it is necessary to specify the shape of the quantizing potential only in the final stage of the numerical procedure of the integration because we have obtained the solutions of the Boltzmann equation in a universal form which holds for any shape of the quantizing potential This is very convenient for a study of the effect of the quantizing potential parameters on the kinetic coefficients. Here we present analytical

calculations of the current–voltage characteristics and the electric-field-dependence of the mean electron energy for some limiting cases in a rectangular QWI.

7.1. Low lattice temperatures

First, we study the case of low lattice temperatures $T_0 < (8m^*s^2W_0)^{1/2}$. The electron current density is equal for the unit cross section of a QWI

$$j_x = -en_0 \frac{\sum_{\kappa_x} v_x(\kappa_x) F(\kappa_x)}{\sum_{\kappa_x} F(\kappa_x)}. \tag{7.1}$$

Substitution of $F(\kappa_x)$ from equation (2.3) into equation (7.1) gives

$$j_x = 2\frac{e^2}{m^*} n_0 E_x \frac{\int_0^\infty \xi_\parallel^{1/2} \tau(\xi_\parallel)(\mathrm{d}F_0(\xi_\parallel)/\mathrm{d}\xi_\parallel)\,\mathrm{d}\xi_\parallel}{\int_0^\infty F_0(\xi_\parallel)(\xi_\parallel)\xi_\parallel^{1/2}\,\mathrm{d}\xi_\parallel} \tag{7.2}$$

where the momentum relaxation time $\tau(\xi_\parallel)$ is given in equation (2.4).

7.1.1. Warm electrons. The distribution function for the warm electrons is given in equations (2.3) and (3.24). Making use of $F_0(\xi_\parallel)$ from equation (3.24), we obtain after integration

$$j_x = en_0\mu_e(\varepsilon_E)E_x. \tag{7.3}$$

Here the mobility of the warm electrons is given by

$$\mu_e(\varepsilon_E) = \mu_0(1 - \beta_0\varepsilon_E^2) \tag{7.4}$$

where β_0 is a general temperature-independent coefficient numerically equal to 0.31 and μ_0 is the low-field mobility of 1D electrons at low temperature, namely

$$\mu_0 = \alpha_0 \frac{e\tau_0}{m^*} \tag{7.5}$$

where α_0 is a general numerical coefficient equal to 0.27.

The expressions in equations (7.3) and (7.4) show that the current–voltage characteristics obey a sub-linear dependence on the electric field. The mean electron energy is

$$\bar{\varepsilon}_\parallel = \frac{\sum_{\kappa_x} \varepsilon_\parallel(\kappa_x) F(\kappa_x)}{\sum_{\kappa_x} F(\kappa_x)} = T_0 \frac{\int_0^\infty F_0(\xi_\parallel)\xi_\parallel^{1/2}\,\mathrm{d}\xi_\parallel}{\int_0^\infty F_0(\xi_\parallel)\xi_\parallel^{-1/2}\,\mathrm{d}\xi_\parallel}. \tag{7.6}$$

Using $F_0(\xi_\parallel)$ from equation (3.24) we obtain the expression for the mean energy of the warm electrons

$$\bar{\varepsilon}_\parallel = \frac{T_0}{2}(1 + \gamma_0\varepsilon_E^2) \tag{7.7}$$

where γ_0 is a general numerical coefficient equal to 0.10. The expressions for the coefficients α_0, β_0 and γ_0 are given in appendix C.

7.1.2. Streaming. Let us investigate the electron–acoustic-phonon streaming regime. The distribution function for this case is given in equation (4.15). Substitution of $F(\kappa_x)$ from equation (4.15) into equation (7.1) gives

$$j_x = en_0\left(\frac{2T_0}{m^*}\right)^{1/2}\frac{E_x}{|E_x|}\frac{\displaystyle\int_0^\infty F_0(\xi_\parallel)\,\mathrm{d}\xi_\parallel}{\displaystyle\int_0^\infty F_0(\xi_\parallel)\xi_\parallel^{-1/2}\,\mathrm{d}\xi_\parallel} = en_0\left(\frac{2T_0}{m^*}\right)^{1/2}\frac{\Gamma(1/3)}{\Gamma(1/6)}\left(\frac{45}{16}\frac{|E_x|}{E_c}\right)^{1/6}\frac{E_x}{|E_x|}. \tag{7.8}$$

The mean electron energy for this case is

$$\bar{\varepsilon}_\parallel = T_0\frac{\Gamma(1/2)}{\Gamma(1/6)}\left(\frac{45}{16}\frac{|E_x|}{E_c}\right)^{1/3}. \tag{7.9}$$

Note that here j_x and $\bar{\varepsilon}_\parallel$ depend neither on the shape of the quantizing potential nor on the lattice temperature T_0 (the last is obvious if we take into account the defintiions of E_c in equation (2.9) and τ_0 in equation (2.5)).

It follows from equation (7.8) that the electron mobility μ_e is a decreasing function of the electric field in the streaming regime:

$$\mu_e = \frac{j_x}{en_0E_x} = \left(\frac{2T_0}{m^*}\right)^{1/2}\frac{\Gamma(1/3)}{\Gamma(1/6)}\left(\frac{45}{16}\frac{|E_x|}{E_c}\right)^{1/6}\frac{1}{|E_x|} \propto |E_x|^{-5/6}. \tag{7.10}$$

The physical explanation of this dependence is that the scattering rate increases with electron energy, $\tau^{-1}(\xi_\parallel) \sim \xi_\parallel^{5/2}$, in accordance with equation (4.14). The electric field pushes electrons into the high-energy region, where $\tau(\xi_\parallel)$ rapidly decreases, resulting in decreasing electron mobility.

Comparison of the electric-field-dependence of the electron current and the mean electron energy $\bar{\varepsilon}_\parallel$ of equations (7.8) and (7.7), $j_x \propto |E_x|^{1/6}$ and $\bar{\varepsilon}_\parallel \propto |E_x|^{1/3}$, with the corresponding results obtained in [14] for the streaming regime, under which $j_x \propto |E_x|^{1/5}$ and $\bar{\varepsilon}_\parallel \propto |E_x|^{2/5}$, shows that the dependences are different, as was pointed out in section 4. The distinctions arise due to the different forms of the energy-dependences of the scattering rates. The authors of [14] performed a Monte Carlo simulation of the streaming regime together with a qualitative description of the electron dynamics without solving the Boltzmann equation. Moreover, they used a semi-empirical parabolic dependence for the scattering rate in the form

$$\tau^{-1}(\varepsilon_\parallel) = \Lambda\varepsilon_\parallel^2 \tag{7.11}$$

where Λ is a constant. Rigorous kinetic analysis gives us the dependence $\tau^{-1}(\xi_\parallel) = (16/15)\tau_0^{-1}\xi_\parallel^{5/2}$ in equation (4.14). Note that this scattering rate depends neither on the cross section of a QWI nor on the lattice temperature T_0.

The expression for the mean electron energy equation (7.9) allows one to determine the range of the electric fields within which the streaming regime occurs. On combining equations (2.1), (4.1) and (7.9) we obtain

$$E_c \ll E_x \ll \frac{16}{45}\left(\frac{\Gamma(1/6)}{\Gamma(1/2)}\right)^3\left(\frac{(8m^*s^2W_0)^{1/2}}{T_0}\right)^3 E_c. \tag{7.12}$$

7.1.3. The ohmic hot-electron regime. With increasing electric field the condition symbolized by the right-hand side of equation (7.12) is broken. The electron–acoustic-phonon interaction becomes quasi-elastic. The distribution function for this case is given in equation (5.8).

The analytical calculations of the kinetic coefficients can be performed for electric fields high enough for it to be possible to omit the coefficient $C(\varepsilon'_\parallel)$ in equation (5.9). Then for the rectangular QWI we obtain from equation (5.8)

$$F_0(\xi_\parallel) = B_0 \exp\left(-\xi_\parallel^{1/2} \frac{E_c'^2}{E_x^2}\right) \tag{7.13}$$

in agreement with the asymptotic expression in equation (5.10). Here we have introduced the notation

$$E'_c = \frac{9\sqrt{2}}{\pi} \frac{W_\perp}{T_0} \left(\frac{2m^* s^2}{T_0}\right)^{7/4} E_c. \tag{7.14}$$

Under these conditions the momentum relaxation time $\tau(\xi_\parallel)$ is that given in equation (5.6). Since $\tau(\xi_\parallel)$ does not depend on $\xi_\parallel$, equation (7.10) gives after the integration by parts

$$j_x = en_0 \mu_e E_x \tag{7.15}$$

where μ_e is the electron mobility

$$\mu_e = \left(\frac{\pi}{3}\right)^2 \frac{e\rho\hbar^4 s}{(m^*)^3 \Xi_a^2 W_\perp}. \tag{7.16}$$

It follows from equations (7.15) and (7.16) that the current–voltage characteristic obeys a linear dependence on the electric field. This is the 'second ohmic regime'. The first occurs at very small electric field $|E_x| \ll E_c$, for which the symmetrical distribution function has the Maxwell–Boltzmann form. With increasing electric field E_x the electron mobility μ_e decreases following the parabolic law for warm electrons and $\mu_e \propto |E_x|^{-5/6}$ for the hot electrons (see equations (7.4) and (7.10)), reaching a minimum value at the electric field

$$E_x \simeq \left(\frac{(8m^* s^2 W_0)^{1/2}}{T_0}\right)^3 E_c \tag{7.17}$$

(see equation (7.12)). After that there is some increase in μ_e with E_x (this because $\tau(\varepsilon_\parallel)$ increases here with $\varepsilon_\parallel$ in accordance with equation (5.4)) and, finally, μ_e saturates (see equation (7.16)) and the 'second ohmic regime' occurs.

For the mean electron energy we obtain from equations (7.6) and (7.13) the following expression:

$$\bar{\varepsilon}_\parallel = 2T_0 \left(\frac{E_x}{E'_c}\right)^4. \tag{7.18}$$

The strong electric-field-dependence of $\bar{\varepsilon}_\parallel$, as $\bar{\varepsilon}_\parallel \propto E_x^4$, means that, with increasing E_x, the electrons penetrate very rapidly into the high-energy region. The simplified one-sub-band model is broken and it is necessary to take into account the upper sub-bands or interaction with optical phonons. Hence, the range of the electric fields within which the dependence from equations (7.15) and (7.18) applies is quite narrow.

7.2. High-lattice temperatures

To compare with low-lattice-temperature case, discussed above, with the case of a high lattice temperature, let us calculate the kinetic coefficients of the 1D electron gas under the condition $T_0 > (8m^*s^2W_0)^{1/2}$. It was shown in section 6 that a runaway effect for hot 1D electrons occurs at high lattice temperature and hence that it is necessary to take into account some additional mechanisms of electron energy relaxation. The interaction of the 1D electrons with optical phonons as well as with acoustic phonons was studied in [18] via Monte Carl simulation and analytically in [19]. The field-dependence of the electron mobility in a QWI was studied in [15] for the case of transition of the electrons to the classical range of energies, ignoring the interaction with optical phonons. As was shown in [19], this situation can be realized in thick QWIs. Here we will follow the method suggested in [15] and present expressions for the electron mobility and the mean electron energy in a 1D rectangular QWI with square cross section, ignoring optical phonons.

We obtain for the electron mobility

$$\mu_e = \mu_e^{3D}\frac{\pi}{3}\frac{T_0}{W_\perp^0}\frac{1+\left[\dfrac{9}{16}\left(\dfrac{2}{\pi}\right)^{1/2}\right]\varepsilon_E'\left(\dfrac{W_\perp^0}{T_0}\right)^{1/2}\exp\left(-\dfrac{1}{\pi\varepsilon_E'}\dfrac{W_0}{T_0}\right)}{1+\left[\dfrac{3\Gamma(3/4)}{4\pi}\left(\dfrac{9}{8}\right)^{1/4}\varepsilon_E'^{3/2}\left(\dfrac{T_0}{W_\perp^0}\right)^{1/4}\exp\left(-\dfrac{1}{\pi\varepsilon_E'}\dfrac{W_\perp^0}{T_0}\right)\right]} \tag{7.19}$$

where

$$\mu_e = \mu_e^{3D} = \frac{4}{3}\frac{e\lambda_a}{(2\pi m^*T_0)^{1/2}} \tag{7.20}$$

is the low-field electron mobility in the 3D case. When the electric field increases, the relative contribution of the second term in the numerator of equation (7.19) becomes more important than is the contribution of the second term in the denominator (this is because $W_\perp^0 \gg T_0$). Hence, we can drop the second term in the denominator and obtain the following asymptotic approximation for the shape of the electric-field-dependence:

$$\mu_e \propto \mu_e^{3D}\frac{\pi}{3}\frac{T_0}{W_\perp^0}\left[1+|E_x|\exp\left(-\frac{1}{|E_x|}\right)\right]. \tag{7.21}$$

Equation (7.21) describes a steep rise of μ_e with E_x for the electric field range $1 \ll \varepsilon_E' \leqslant 2W_\perp^0/T_0$. This means that the current–voltage characteristics obey a super-linear dependence on electric field. The physical reason for this is the rise in relaxation time with heating of the electrons (see equation (6.5)).

At high electric fields, for which both field-dependent terms in equation (7.19) are important, we can drop the units in the numerator and denominator and obtain

$$\mu_e \propto \mu_e^{3D}\frac{1}{|E_x|^{1/2}}. \tag{7.22}$$

This dependence corresponds to the situation of a 3D hot electron gas in which the transition of electrons into the classical range of the energies occurs.

For the mean electron energy we obtained the following expression:

$$\bar{\varepsilon}_\parallel = \frac{T_0}{2}\frac{1+\left[\left(\dfrac{3}{2\pi}\right)^2\left(\dfrac{9}{2}\right)^{1/4}\Gamma(5/4)\right]\varepsilon_E'^{5/2}\left(\dfrac{W_\perp^0}{T_0}\right)^{1/4}\exp\left(-\dfrac{1}{\pi\varepsilon_E'}\dfrac{W_\perp^0}{T_0}\right)}{1+\left[\dfrac{3\Gamma(3/4)}{4\pi}\left(\dfrac{9}{8}\right)^{1/4}\right]\varepsilon_E'^{3/2}\left(\dfrac{T_0}{W_\perp^0}\right)^{1/4}\exp\left(-\dfrac{1}{\pi\varepsilon_E'}\dfrac{W_\perp^0}{T_0}\right)}. \tag{7.23}$$

The same situation applies here with the relative contributions of the second terms in the numerator and denominator as it was in equation (7.19).

For the electric fields, when we can omit the second term in the denominator, we obtain

$$\bar{\varepsilon} \propto \frac{T_0}{2}\left[1 + |E_x|^{5/2}\exp\left(-\frac{1}{|E_x|}\right)\right]. \tag{7.24}$$

This dependence corresponds to a very steep rise of $\bar{\varepsilon}$ with E_x.

For even higher electric fields we can omit units in equation (7.18) and so we obtain

$$\bar{\varepsilon} \propto T_0|E_x|. \tag{7.25}$$

This much slower increase in $\bar{\varepsilon}$ with E_x corresponds to the 3D hot-electron gas.

7.3. Comparison of temperature regimes

A comparison of the macroscopic characteristics of the 1D non-equilibrium electron gas at low and high lattice temperatures derived above shows us the qualitatively different behaviour of their electric-field-dependences. The general physical reason for this is the existence of the characteristic energy $\varepsilon_c = (8m^*s^2W_0)^{1/2}$ for a 1D electron gas in a QWI interacting with 3D acoustic phonons.

For a high lattice temperature, one at which $T_0 > \varepsilon_c$, the electron–acoustic-phonon interaction has a quasi-elastic character at all values of electric field. The energy loss due to this interaction is not strong enough to ensure a balance of the energy gained by the electron system from the external electric field because the scattering rate decreases when the electron energy increases. As a result the electrons penetrate very rapidly into the high-energy region, resulting in a steep increase in electron mobility (super-linear current–voltage characteristics) and mean energy (see equations (7.21) and (7.24)).

For a low lattice temperature, one at which $T_0 < \varepsilon_c$, the electron–acoustic-phonon interaction has a strongly inelastic character within a wide electric field range. As a result the energy loss mechanism is very effective and the momentum relaxation time decreases when the electron energy increase. This results in a slow increase in the mean electron energy and a decrease in the electron mobility (sub-linear current–voltage characteristics) when the electric field increases (see equations (7.9) and (7.4) and (7.10), respectively). For the higher electric fields, at which the electrons penetrate into the high-energy region, the electron–acoustic-phonon interaction becomes quasi-elastic. The scattering rate here saturates with increasing electron energy. As a result a normalized steady state electron distribution exists and we obtain the 'second ohmic regime' in the current–voltage characteristic within a narrow range of the electric fields.

8. Conclusion

In this paper, and in the previous papers I and II, we have developed a general kinetic theory of the non-equilibrium electrons in a 1D QWI interacting with the bulk acoustic phonons for an arbitrary shape of the quantizing potential. Due to the size quantization of the electron motion a new characteristic energy $\varepsilon_c = (8m^*s^2W_0)^{1/2}$ arises in the theory (W_0 is a quantum energy of the ground electron state). For a GaAs rectangular QWI with $L_y = L_z \equiv L_\perp = 50$ Å, $m^* = 0.07m_e$ and $s = 5.14 \times 10^5$ cm s^{-1}, we obtain $W_0 = \pi^2\hbar^2/(m^*L_\perp^2) = 430$ meV, $m^*s^2 = 0.01$ meV and $\varepsilon_c = 6$ meV $\simeq 70$ K. As one can see, ε_c corresponds to the actual electron energy region.

The electron kinetics in a 1D QWI undergoes a crucial change depending on the relationship between the characteristic energy ε_c and the lattice temperature T_0. We investigated here all possible cases that are of physical interest and amenable to experimental investigation and we derived the expressions for the collision operator for these cases.

For high lattice temperature, at which $T_0 \gtrsim \varepsilon_c$, the electron–acoustic-phonon interaction has a quasi-elastic character for the majority of electrons. As a result, for the antisymmetrical collision operator we can apply the electron momentum relaxation time approximation, in which the energy-dependence of the relaxation time is defined by the energy-dependence of the electron density of states of equation (6.5). The symmetrical collision operator, due to the relative smallness of the change in the electron energy in each event, can be transformed into the differential Fokker–Planck form. The corresponding Boltzmann equation represents the continuity equation in energy space, equation (6.11). From the mathematical point of view this is a second-order differential equation and it can be easily solved, resulting in the distribution function in equation (6.15). The characteristic electric field E_0 for this case is defined in equation (6.20) and it can be presented in the form $E_0 = 3\sqrt{3}\varepsilon_c/(4\pi\sqrt{2}e\lambda_a)$. The electrons are hot if $E_x \gg E_0$. For $T_0 = 80$ K and GaAs parameters, $\Xi_a = 7$ eV, $\rho = 5.31$ g cm^{-1}, we obtain $\lambda_a = \pi\hbar^4\rho s^2/[\Xi_a^2(m^*)^2T_0] = 0.95 \times 10^{-3}$ cm and $E_0 = 16.5$ V cm^{-1}, for the same geometric size of a QWI as above.

For low lattice temperature, at which $T_0 \ll \varepsilon_c$, the behaviour of the electron system is more complicated compared with that in the high-lattice-temperature case. Within the wide electric field range the electron–acoustic-phonon collisions have a strongly inelastic character. The condition need to derive the Fokker–Planck form for the collision operator is violated and consequently the symmetrical part of the electron–acoustic-phonon collision operator has the integral form, equation (5.6) in paper II. For the antisymmetrical collision operator it is possible to use the momentum relaxation approximation with the relaxation time given by equation (2.4). As a result the corresponding Boltzmann equation has an integro-differential form. For the low electric field $E_x < E_c$ this equation can be transformed into a pure integral equation but not into one of Fredholm's type. We solved this equation analytically and obtained the distribution function for the warm electrons. For the high electric field defined in equation (4.7), $E_c \ll E_x \ll E_c^*$, the electron distribution function has a strongly anisotropic shape, equation (4.15), corresponding to the streaming regime. The characteristic field $E_c^* \cong 5m^*\Xi_a^2\varepsilon_c^3/(2\pi e\rho\hbar^4 s^4)$ is given by the right-hand side of the inequality in equation (4.7) and the other characteristic field $E_c = m^*\Xi_a^2T_0^3/(4\pi e\rho\hbar^4 s^4)$ is given in equation (2.9). For the same parameter of a QWI as above and the lattice temperature $T_0 = 4.2$ K we obtain $E_c \simeq 0.2$ V cm^{-1} and $E_c^* \simeq 8.6 \times 10^3$ V cm^{-1}. Note that the electric field E_c^* does not depend on T_0. For electric field $E_x \geqslant E_c^*$ the electrons penetrate in the high-energy region, the electron–phonon interaction becomes a quasi-elastic one and the distribution function has a quasi-isotropic form, equations (2.3) and (5.8). We have also calculated the some macroscopic characteristic of the non-equilibrium 1D electron gas in a QWI for all the above situations and presented here their electric-field-dependences.

In this paper we have taken into account only the interaction with acoustic phonons which is responsible for the relaxation both of the electron momentum and of the electron energy. However, the effect of elastic scattering due to interface roughness or charged background and remote impurities can easily be incorporated. The same can be done for the strong inelastic interaction with optical phonons [19] if the lattice temperature is small compared with the optical phonon energy (for GaAs this is 36.6 meV).

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Appendix A. Derivation of the integration constant B_0 in equation (2.21)

The goal of this appendix is to derive the expression for the integration constant B_0 in equation (2.21). We will use for this the second boundary condition from equation (2.24).

On substituting $F_0(\xi_\parallel)$ from equation (2.21) into equation (2.24) we obtain

$$\lim_{\xi_\parallel\to\infty} F_0(\xi_\parallel) = A_0 \lim_{\xi_\parallel\to\infty} \tilde{F}_0^-(\xi_\parallel) + B_0 \lim_{\xi_\parallel\to\infty} \tilde{F}_0^+(\xi_\parallel) + \lim_{\xi_\parallel\to\infty} \int_0^\infty H(\omega,\xi_\parallel) F_0(\omega)\,\mathrm{d}\omega. \tag{A1}$$

The function $\Psi_1^{-1}(\xi_\parallel)$ determines the energy-dependences of $\tilde{F}_0^\pm(\xi_\parallel)$ and, in accordance with equation (2.6), $\Psi_1^{-1}(\xi_\parallel)$ increases with energy $\xi_\parallel$. Taking into account equation (2.19) this gives

$$\lim_{\xi_\parallel\to\infty} \tilde{F}_0^-(\xi_\parallel) = 0.$$

Note that it would be incorrect to put $B_0 = 0$ due to the infinitely increasing $\lim_{\xi_\parallel\to\infty} \tilde{F}_0^+(\xi_\parallel)$ in equation (A1). To see this let us substitute $H(\omega,\xi_\parallel)$ from equation (2.22) into equation (A1) and re-write equation (A1) in the form

$$\begin{aligned}\lim_{\xi_\parallel\to\infty} \tilde{F}_0^+(\xi_\parallel)\Bigg(B_0 - \lim_{\xi_\parallel\to\infty} \frac{1}{2\varepsilon_E} \int_0^\infty \mathrm{d}\omega\, F_0(\omega) \int_0^{\xi_\parallel} K_1(\omega,\xi_\parallel')\tilde{F}_0^+(\xi_\parallel')\,\mathrm{d}\xi_\parallel' \Bigg) \\ + \lim_{\xi_\parallel\to\infty} \tilde{F}_0^+(\xi_\parallel) \frac{1}{2\varepsilon_E} \int_0^\infty \mathrm{d}\omega\, F_0(\omega) \int_0^{\xi_\parallel} K_1(\omega,\xi_\parallel')\tilde{F}_0^-(\xi_\parallel')\,\mathrm{d}\xi_\parallel' = 0.\end{aligned} \tag{A2}$$

To calculate the limit in the second term in equation (A2) it is necessary to change the order of the operation of limit and integration. It gives for the second term

$$\begin{aligned}&\lim_{\xi_\parallel\to\infty} \tilde{F}_0^+(\xi_\parallel) \frac{1}{2\varepsilon_E} \int_0^\infty \mathrm{d}\omega\, F_0(\omega) \int_0^{\xi_\parallel} K_1(\omega,\xi_\parallel')\tilde{F}_0^-(\xi_\parallel)\,\mathrm{d}\xi_\parallel' \\ &\quad= \frac{1}{2\varepsilon_E} \int_0^\infty \mathrm{d}\omega\, F_0(\omega) \lim_{\xi_\parallel\to\infty} \left(\tilde{F}_0^+(\xi_\parallel') \int_0^{\xi_\parallel} K_1(\omega,\xi_\parallel')\tilde{F}_0^-(\xi_\parallel')\,\mathrm{d}\xi_\parallel' \right) \\ &\quad= \frac{1}{2\varepsilon_E} \int_0^\infty \mathrm{d}\omega\, F_0(\omega) \lim_{\xi_\parallel\to\infty} \left(\tilde{F}_0^+(\xi_\parallel') \int_0^{\xi_\parallel} \frac{(\xi_\parallel'-\omega)^2}{(\xi_\parallel'\omega)^{1/2}} \left| \frac{1}{\mathrm{e}^{\xi_\parallel'} - \mathrm{e}^{\omega}} \right| \tilde{F}_0^-(\xi_\parallel')\,\mathrm{d}\xi_\parallel' \right)\end{aligned} \tag{A3}$$

where we have used the expression for $K_1(\omega,\xi_\parallel')$ from equation (2.12).

Inasmuch as the function $\tilde{F}_0^+(\xi_\parallel)$ is a growing function of $\xi_\parallel$ it is evident that the main contribution to the integral in parentheses comes from the region with very large values of $\xi_\parallel'$. Then we obtain for the limit the following estimation:

$$\begin{aligned}&\lim_{\xi_\parallel\to\infty} \left(\tilde{F}_0^+(\xi_\parallel) \int_0^{\xi_\parallel} \frac{(\xi_\parallel'-\omega)^2}{(\xi_\parallel'\omega)^{1/2}} \left| \frac{1}{\mathrm{e}^{\xi_\parallel'} - \mathrm{e}^{\omega}} \right| \tilde{F}_0^-(\xi_\parallel')\,\mathrm{d}\xi_\parallel' \right) \\ &\quad\simeq \frac{1}{\omega^{1/2}} \lim_{\xi_\parallel\to\infty} \left(\tilde{F}_0^+(\xi_\parallel) \int_0^{\xi_\parallel} \xi_\parallel'^{3/2}\, \mathrm{e}^{-\xi_\parallel'} \tilde{F}_0^-(\xi_\parallel')\,\mathrm{d}\xi_\parallel' \right)\end{aligned}$$

$$< \frac{1}{\omega^{1/2}} \lim_{\xi_\parallel \to \infty} [\tilde{F}_0^+(\xi_\parallel)(\tilde{F}_0^-(\xi_\parallel)\xi_\parallel^{3/2}\,\mathrm{e}^{-\xi_\parallel})\xi_\parallel]$$

$$= \frac{1}{\omega^{1/2}} \lim_{\xi_\parallel \to \infty} (\xi_\parallel^{5/2}\,\mathrm{e}^{-\xi_\parallel}) = 0. \tag{A4}$$

The means that the limit in equation (A3) is equal to zero as well and the same is true for the last term in equation (A2). Then the first term in equation (A2) can be put equal to zero if and only if the expression in parentheses is equal to zero. This gives the following expression for B_0:

$$B_0 = \frac{1}{2\varepsilon_E} \int_0^\infty \int_0^\infty K_1(\omega, \xi_\parallel')\tilde{F}_0^-(\xi_\parallel')F_0(\omega)\,\mathrm{d}\xi_\parallel'\,\mathrm{d}\omega \tag{A5}$$

which is used in equation (2.25).

Appendix B. Application of the Laplace method to the transformation of the kernel in the integral equation (3.8)

Let us transform the function $H_1(\omega, \xi_\parallel)$ in equation (2.28). Since the function $K_1(\omega, \xi_\parallel')$ has a special point $\xi_\parallel' = \omega$, see figure 4, where the derivative $\mathrm{d}K_1(\omega, \xi_\parallel')/\mathrm{d}\xi_\parallel'$ suffers a discontinuity, we present $H_1(\omega, (\xi_\parallel)$ in the form

$$H_1(\omega, \xi_\parallel') = \theta(\xi_\parallel - \omega)\left[\int_0^\omega K_1'(\omega, \xi_\parallel') \exp\left(-\frac{\varphi(\xi_\parallel')}{\varepsilon_E}\right) \mathrm{d}\xi_\parallel' + \int_\omega^{\xi_\parallel} K_1''(\omega, \xi_\parallel') \exp\left(-\frac{\varphi(\xi_\parallel')}{\varepsilon_E}\right) \mathrm{d}\xi_\parallel'\right] + \theta(\omega - \xi_\parallel) \int_0^{\xi_\parallel} K_1'(\omega, \xi_\parallel') \exp\left(-\frac{\varphi(\xi_\parallel')}{\varepsilon_E}\right) \mathrm{d}\xi_\parallel' \tag{B1}$$

where $\theta(z)$ is the Heaviside step function and

$$K_1(\omega, \xi_\parallel') = \begin{cases} K_1'(\omega, \xi_\parallel') & \xi_\parallel' \leqslant \omega \\ K_1''(\omega, \xi_\parallel') & \xi_\parallel' \geqslant \omega \end{cases} \tag{B2}$$

$$\varphi(\xi_\parallel') = \int_{\xi_\parallel'}^{\xi_\parallel} \frac{\mathrm{d}\xi_\parallel'}{\Psi_1(\xi_\parallel')} = J(\xi_\parallel) - J(\xi_\parallel'). \tag{B3}$$

The functions $K_1'(\omega, \xi_\parallel')$ and $K_1''(\omega, \xi_\parallel')$ are the smooth functions of $\xi_\parallel'$ without disruption of the derivatives. Under the integrals in equation (B1) we have products of the slowly changing functions $K_1'(\omega, \xi_\parallel')$ and $K_1''(\omega, \xi_\parallel')$ and rapidly changing $\exp(-\varphi(\xi_\parallel')/\varepsilon_E)$.

In accordance with the Laplace method [10], to calculate the integrals it is necessary to perform the integration by parts in equation (A1) by introducing new variables:

$$u = \frac{\mathrm{d}\varphi(\xi_\parallel')}{\mathrm{d}\xi_\parallel'}[K_1'(\omega, \xi_\parallel'), K_1''(\omega, \xi_\parallel')]$$
$$\mathrm{d}v = \exp\left(-\frac{\varphi(\xi_\parallel')}{\varepsilon_E}\right) \frac{\mathrm{d}\varphi(\xi_\parallel')}{\mathrm{d}\xi_\parallel'}\,\mathrm{d}\xi_\parallel'. \tag{B4}$$

Taking into account the inequality from equation (3.7), we obtain after some transformations during the process of the integration by parts the following expression for $H_1(\omega, \xi_\parallel)$:

$$H_1(\omega, \xi_\parallel) = \varepsilon_E \left(\frac{\mathrm{d}J(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right)^{-1} K_1(\omega, \xi_\parallel) - \varepsilon_E \int_0^{\xi_\parallel} \exp\left(-\frac{\varphi(\xi_\parallel')}{\varepsilon_E}\right) \frac{\mathrm{d}}{\mathrm{d}\xi_\parallel'}[K_1(\omega, \xi_\parallel')\Psi_1(\xi_\parallel')]\,\mathrm{d}\xi_\parallel'. \tag{B5}$$

By repeating the same algorithm twice we obtain the expression

$$H_1(\omega,\xi_\parallel)=\varepsilon_E\left(\frac{\mathrm{d}J(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right)^{-1}K_1(\omega,\xi_\parallel)$$
$$-\varepsilon_E^2\left(\frac{\mathrm{d}J(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left[\left(\frac{\mathrm{d}J(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right)^{-1}K_1(\omega,\xi_\parallel)\right]+\varepsilon_E^3\left(\frac{\mathrm{d}J(\xi_\parallel)}{\mathrm{d}\xi_\parallel}\right)^{-1}$$
$$\times\int_0^{\xi_\parallel}\exp\left(-\frac{\varphi(\xi_\parallel')}{\varepsilon_E}\right)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel'}\left[\Psi_1(\xi_\parallel')\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel'}\left(\Psi_1(\xi_\parallel')\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel'}[\Psi_1(\xi_\parallel')K_1(\omega,\xi_\parallel')]\right)\right]\mathrm{d}\xi_\parallel'. \tag{B6}$$

As one can see from equation (A6), each foregoing term is smaller than each following one by a factor $\varepsilon_E/J(\xi_\parallel)\ll 1$. By this is meant that equation (A6) is a presentation of $H_1(\omega,\xi_\parallel)$ in the form of an asymptotic expansion in the parameter $\varepsilon_E/J(\xi_\parallel)$. The next term in equation (A6) after the integration by parts will have the order $(\varepsilon_E/J(\xi_\parallel))^4\ll 1$. Hence, within a factor $(\varepsilon_E/J(\xi_\parallel))^3$ we obtain the expression for $H_1(\omega,\xi_\parallel)$

$$H_1(\omega,\xi_\parallel)=\varepsilon_E\Psi_1(\xi_\parallel)K_1(\omega,\xi_\parallel)-\varepsilon_E^2\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}[\Psi_1(\xi_\parallel)K_1(\omega,\xi_\parallel)]$$
$$+\varepsilon_E^3\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left(\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}[\Psi_1(\xi_\parallel)K_1(\omega,\xi_\parallel)]\right) \tag{B7}$$

in which equation (3.6) was taken into account namely that $\mathrm{d}J(\xi_\parallel)/\mathrm{d}\xi_\parallel=\Psi_1^{-1}(\xi_\parallel)$.

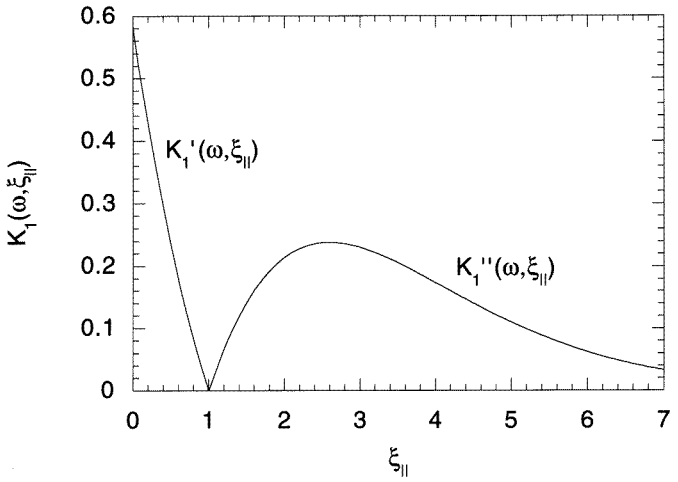

Figure 4. The dependence of $K_1(\omega,\xi_\parallel)$ on $\xi_\parallel$ for the particular case $\omega=1$.

By applying the same algorithm to $H_2(\omega,\xi_\parallel)$ from equation (2.29) we obtain the expression

$$H_2(\omega,\xi_\parallel)=\varepsilon_E\Psi_1(\xi_\parallel)K_1(\omega,\xi_\parallel)+\varepsilon_E^2\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}[\Psi_1(\xi_\parallel)K_1(\omega,\xi_\parallel)]$$
$$+\varepsilon_E^3\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}\left(\Psi_1(\xi_\parallel)\frac{\mathrm{d}}{\mathrm{d}\xi_\parallel}[\Psi_1(\xi_\parallel)K_1(\omega,\xi_\parallel)]\right). \tag{B8}$$

The expressions for $H_{1,2}(\omega,\xi_\parallel)$ are used in equation (3.11).

Appendix C. Coefficients α_0, β_0 and γ_0 for warm electrons

On substituting the distribution function $F_0(\xi_\parallel)$ from equation (3.24) and $\tau(\xi_\parallel)$ from equation (2.4) into equations (7.2) and (7.6) we obtain for the current density and the mean electron energy, respectively

$$j_x = 2\frac{e^2\tau_0}{m^*} n_0 E_x \frac{B_0 + \varepsilon_E^2 B_1}{A_0 + \varepsilon_E^2 A_1} \tag{C1}$$

$$\bar{\varepsilon}_\parallel = T_0 \frac{C_0 + \varepsilon_E^2 C_1}{A_0 + \varepsilon_E^2 A_1} \tag{C2}$$

where

$$A_0 = \int_0^\infty \xi_\parallel^{-1/2} \mathrm{e}^{-\xi_\parallel}\, \mathrm{d}\xi_\parallel = \sqrt{\pi} \tag{C3}$$

$$A_1 = \int_0^\infty \xi_\parallel^{-1/2} g_0(\xi_\parallel)\, \mathrm{e}^{-\xi_\parallel}\, \mathrm{d}\xi_\parallel \tag{C4}$$

$$B_0 = \int_0^\infty \Psi_1(\xi_\parallel)\, \mathrm{e}^{-\xi_\parallel}\, \mathrm{d}\xi_\parallel \tag{C5}$$

$$B_1 = \int_0^\infty \frac{\mathrm{d}\Psi_1(\xi_\parallel)}{\mathrm{d}\xi_\parallel} g_0(\xi_\parallel)\, \mathrm{e}^{-\xi_\parallel}\, \mathrm{d}\xi_\parallel \tag{C6}$$

$$C_0 = \int_0^\infty \xi_\parallel^{-1/2}\, \mathrm{e}^{-\xi_\parallel}\, \mathrm{d}\xi_\parallel = \frac{\sqrt{\pi}}{2} \tag{C7}$$

$$C_1 = \int_0^\infty \xi_\parallel^{-1/2} g_0(\xi_\parallel)\, \mathrm{e}^{-\xi_\parallel}\, \mathrm{d}\xi_\parallel. \tag{C8}$$

The functions $\Psi_1(\xi_\parallel)$ and $g_0(\xi_\parallel)$ are given in equations (2.11) and (3.20), respectively. Taking into account that for the warm electrons $\varepsilon_E^2 \ll 1$, we obtain from equations (C1) and (C2)

$$j_x = \alpha_0 \frac{e^2\tau_0}{m^*} n_0 E_x (1 - \beta_0 \varepsilon_E^2) \tag{C9}$$

$$\bar{\varepsilon}_\parallel = \frac{T_0}{2}(1 + \gamma_0 \varepsilon_E^2) \tag{C10}$$

where

$$\alpha_0 = \frac{2}{\sqrt{\pi}} B_0 \qquad \beta_0 = \frac{1}{\sqrt{\pi}} \left(A_1 - \sqrt{\pi}\frac{B_1}{B_0} \right) \qquad \gamma_0 = \frac{1}{\sqrt{\pi}}(2C_1 - A_1).$$

Numerical integration in equations (A4)–(A6) and (A8) results in the following values: $A_1 = -0.0584$, $B_0 = 0.2415$, $B_1 = -0.0837$ and $C_1 = 0.0512$. This gives $\alpha_0 = 0.27$, $\beta_0 = 0.31$ and $\gamma_0 = 0.10$. These values were used in equations (7.4), (7.5) and (7.7).

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